Math 1272: Calculus II
Midterm II Review

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Find the tangent line to the parametric curve

$$x = t^2, \quad y = t\ln(t) - t$$

at $t = 2$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t\ln(t) - t)}{\frac{d}{dt}t^2}$$

$$= \frac{\ln(t) + \frac{t}{t} - 1}{2t} = \frac{\ln(t)}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{\ln(2)}{4}$$

At $t=2$, $x=4$, $y=2\ln(2)-2$

$$y-2\ln(2)-2=\frac{\ln(2)}{4}(x-4)$$

$$\left(y-y_0=m(x-x_0)\right)$$

$$y=2+2\ln(2)-\ln(2)+\frac{\ln(2)}{4}x$$

$$\boxed{y=2+\ln(2)+\frac{\ln(2)}{4}x}$$

Find the solution of the differential equation

$$y' = \ln(x)y$$

Separable

$$\frac{dy}{dx} = \ln(x)\,y$$

$$\int \frac{dy}{y} = \int \ln(x)\,dx$$

integrate by parts

$u = \ln(x)$
$dv = dx$

$$\ln|y| = x\ln(x) - x + C$$

$$y = \pm e^{x\ln(x) - x + C}$$

$$y = Ae^{x\ln(x) - x}$$

Solve the initial value problem

$$x^2y' + 2xy = \ln x, \quad y(1) = 2.$$

linear integrating factor

$$y' + \frac{2}{x}y = \frac{\ln(x)}{x^2}$$

$$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} \quad (x>0)$$

$$= e^{\ln(x^2)}$$

$$= x^2$$

$$\frac{d}{dx}\left(x^2 y\right) = x^2y' + 2xy = \ln(x)$$

$$x^2 y = x\ln(x) - x + C$$

$$y(1) = 2$$

$$(1)^2 2 = 1\cancel{\ln(1)}^{0} - 1 + C$$

$$2 = -1 + C \implies C = 3$$

$$\boxed{y = \frac{\ln(x)}{x} - \frac{1}{x} + 3}$$

Solve the initial value problem

$$xy' = y + x^2 \sin x, \quad y(\pi) = 0.$$

$$y' - \frac{1}{x} y = x \sin(x)$$

Integrating factor

$$I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)}$$

$$= e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$(x>0)$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{1}{x} y' - \frac{1}{x^2} y$$

$$= \frac{1}{x}\left(y' - \frac{1}{x} y\right)$$

$$= \frac{1}{x} x \sin(x) = \sin(x)$$

$$\frac{1}{x} y = -\cos(x) + C$$

$$y = -x\cos(x) + Cx$$

$$y(\pi) = 0$$

$$0 = -\pi\cos(\pi) + C\pi$$

$$= \pi + C\pi \implies C = -1$$

$$\boxed{y = -x\cos(x) - x}$$

A tank is filled with 100L of a salt-water mixture containing 10kg of dissolved salt. Brine containing 10g/L (1000 g = 1kg) of salt enters the tank at a rate of 1L/hour, while the salt-water solution in the tank drains at a rate of 2L/hour. Write down a differential equation for the amount of salt $S(t)$ in the tank for $t \leq 100$, assuming the salt-water mixture is always kept thoroughly mixed. How much salt is remaining after 10 hours?

$S(0) = 10.$ $\quad \left[1 \frac{L}{hr} \text{ in} - 2 \frac{L}{hr} \text{ out}\right]$

Amount of mixture in tank $= 100 - t$, $t \leq 100$

$$\text{Concentration} = \frac{S(t)}{100-t} \frac{kg}{L}$$

$$\text{Flow out} = \frac{S(t)}{100-t} \frac{kg}{L} \times 2 \frac{L}{hr} = \frac{2S(t)}{100-t}$$

$$\text{Flow in} = \frac{1}{100}\frac{kg}{L} \times 1\frac{L}{hr} = \frac{1}{100}\frac{kg}{hr}$$

$$\boxed{\begin{aligned}\frac{dS}{dt} &= \text{Flow in} - \text{Flow out}\\ &= \frac{1}{100} - \frac{2S(t)}{100-t}\end{aligned}}$$

$$S' + \frac{2}{100-t}S = \frac{1}{100}$$

Integrating factor $I(x) = e^{\int \frac{2}{100-t}\,dt}$

$$= e^{-2\ln(100-t)}$$

$$= e^{\ln\left((100-t)^{-2}\right)}$$

$$= \frac{1}{(100-t)^2}$$

$$\frac{d}{dt}\left(\frac{S(t)}{(100-t)^2}\right) = \frac{S'(t)}{(100-t)^2} + \frac{2S(t)}{(100-t)^3}$$

$$= \frac{1}{(100-t)^2}\left(S'(t) + \frac{2S(t)}{100-t}\right)$$

$$= \frac{1}{100(100-t)^2}$$

$$\frac{S(t)}{(100-t)^2} = \frac{-1}{100(100-t)} + C$$

$$S(t) = C(100-t)^2 - \frac{100-t}{100}$$

$$= C(100-t)^2 + \frac{1}{100}t - 1$$

$$10 = S(0) = C\,100^2 - 1 \longrightarrow C = \frac{11}{10^2}$$

$$S(t) = \frac{11}{10^2}(100-t)^2 + \frac{1}{100}t - 1$$

$$= 11\left(1 - \frac{t}{100}\right)^2 + \frac{t}{100} - 1$$

After 10 hours

$$S(10) = 11\left(1 - \frac{10}{100}\right)^2 + \frac{10}{100} - 1$$

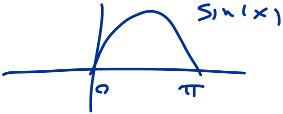
$$= 11(1 - 0.1)^2 + 0.1 - 1$$

$$= 11(0.9)^2 - 0.9$$

$$= 8.01 \text{ kg}$$

Find the area of the region enclosed by one loop of the curve

$$r = 10\sin(\pi\theta).$$



One loop $\quad \sin(\pi\theta) : 0 \longrightarrow 0$

$$\pi\theta : 0 \longrightarrow \pi$$

$$\theta : 0 \longrightarrow 1$$

$$\text{Area} = \int_0^1 \frac{1}{2}\left(10\sin(\pi\theta)\right)^2 d\theta$$

$$= 50\int_0^1 \sin^2(\pi\theta)\, d\theta.$$

$$= 25 \int_0^1 1 - \cos(2\pi\theta)\, d\theta$$

$$= 25 \left(\theta - \frac{1}{2\pi} \sin(2\pi\theta) \right) \Big]_0^1$$

$$= 25$$

small = 10 bees (missing information)

A population of honeybees grows at a rate of 10 bees per hour when the population is small. The hive can contain at most 1000 bees. Use a logistic differential equation to model the honeybee population $P(t)$. If the population starts with 10 honeybee, how many bees will there be after 10 hours.

$$\frac{dP}{dt} \approx kP = 10 \quad \text{when} \quad P = 10 \implies k = 1$$

Logistic

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad k = 1, \quad M = 1000$$

$$\frac{dP}{dt} = P\left(1 - \frac{P}{1000}\right)$$

$P_0 = 10$, $\quad A = \dfrac{M - P_0}{P_0} = \dfrac{1000 - 10}{10} = \dfrac{9990}{10}$

$= 999$

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{1000}{1 + 999e^{-t}}}$$

After 10 hours $\quad P(10) = \dfrac{1000}{1 + 999e^{-10}} \simeq 956.6$

Can leave like this

Find dy/dx and d^2y/dx^2 for the parametric curve

$$x = t^3 + 1, y = t^2 - t.$$

For which values of t is the curve concave upward?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{3t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2t-1}{3t^2}\right)}{3t^2}$$

$$= \frac{3t^2(2) - (2t-1)(6t)}{(3t^2)^3}$$

$$= \frac{6t^2 - 12t^2 + 6t}{27t^6}$$

$$= \frac{-6t^2 + 6t}{27t^6}$$

$$= \frac{6}{27}\left(\frac{1-t}{t^5}\right)$$

$\frac{d^2y}{dx^2} < 0$ $\qquad$ $\frac{d^2y}{dx^2} > 0$ $\qquad$ $\frac{d^2y}{dx^2} < 0$

$t=0$ $\qquad$ concave up $\qquad$ $t=1$

$0 \le t \le 1$

Find the surface area obtained by rotating the curve

$$x = t^3, y = t^2, \quad 0 \leq t \leq 1$$

about the x-axis.

$$S = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi t^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$= \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= \int_0^1 2\pi t^3 \sqrt{9t^2 + 4}\, dt$$

$$= \int_0^1 6\pi t^3 \sqrt{t^2 + \left(\frac{2}{3}\right)^2}\, dt$$

$$t = \frac{2}{3}\tan\theta \qquad dt = \frac{2}{3}\sec^2\theta\, d\theta$$

$$\sqrt{t^2 + \left(\frac{2}{3}\right)^2} = \frac{2}{3}\sqrt{1+\tan^2\theta} = \frac{2}{3}\sec\theta$$

$$= \int_0^{\tan^{-1}\left(\frac{3}{2}\right)} 6\pi \left(\frac{2}{3}\right)^3 \tan^3\theta \sec\theta \, \frac{2}{3}\sec^2\theta\, d\theta$$

$$= 6\pi\left(\frac{2}{3}\right)^4 \int_0^{\tan^{-1}\left(\frac{3}{2}\right)} \tan^3\theta \sec^3\theta\, d\theta$$

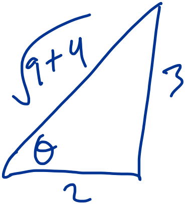
$1+\tan^2=\sec^2$

$u=\sec\theta$, $du=\tan\theta\sec\theta\,d\theta$

$$=6\pi\left(\frac{2}{3}\right)^4\int_1^{\sec\left(\tan^{-1}\left(\frac{3}{2}\right)\right)}(u^2-1)\,u^2\,du$$

$$=6\pi\left(\frac{2}{3}\right)^4\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sec\left(\tan^{-1}\left(\frac{3}{2}\right)\right)}$$

$\tan\theta=\frac{3}{2}$



$$\sec\theta=\frac{1}{\cos\theta}$$

$$=\frac{\sqrt{9+4}}{2}$$

$$\longrightarrow \quad \sec\left(\tan^{-1}\left(\frac{3}{2}\right)\right) = \frac{\sqrt{13}}{2}$$

$$= 6\pi\left(\frac{2}{7}\right)^4\left[\frac{1}{5}\left(\frac{\sqrt{13}}{2}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{13}}{2}\right)^3 - \frac{1}{5} + \frac{1}{3}\right]$$

Note: This problem is a bit complicated for midterm.

Find the arclength of the curve

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leq t \leq 1.$$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^1 6t\sqrt{1 + t^2}\, dt$$

$$u = 1 + t^2$$

$$du = 2t\, dt$$

$$6t\, dt = 3du$$

$$= \int_1^2 3\sqrt{u}\, du$$

$$= 3 \cdot \frac{2}{3} u^{\frac{3}{2}} \Big]_1^2$$

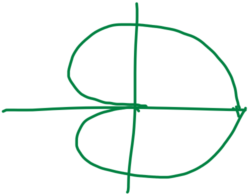
$$= 2\left(2^{3/2} - 1\right)$$

$$= 2\left(2\sqrt{2} - 1\right)$$

$$= 4\sqrt{2} - 2$$

Find the arclength of the polar curve

$$r = 2(1 + \cos\theta).$$



$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4(1+\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4(1+2\cos\theta + \cos^2\theta) + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + \underbrace{4\cos^2\theta + 4\sin^2\theta}_{=4}} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta} \, d\theta$$

$$= \sqrt{8} \int_0^{2\pi} \sqrt{1+\cos\theta} \; \frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}} \, d\theta$$

$$= \sqrt{8} \int_0^{2\pi} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1-\cos\theta}} \, d\theta$$

$$= \sqrt{8} \int_0^{2\pi} \frac{\sqrt{\sin^2\theta}}{\sqrt{1-\cos\theta}}\, d\theta$$

$$= \sqrt{8} \int_0^{2\pi} \frac{|\sin\theta|}{\sqrt{1-\cos\theta}}\, d\theta$$

$$= \sqrt{8} \int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta + \sqrt{8} \int_{\pi}^{2\pi} \frac{-\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta$$

$$u = 1-\cos\theta \qquad du = \sin\theta\, d\theta$$

$$= \sqrt{8} \int_0^{2} \frac{1}{\sqrt{u}}\, du - \sqrt{8} \int_2^{0} \frac{1}{\sqrt{u}}\, du$$

$$= 2\sqrt{8}\int_0^2 \frac{1}{\sqrt{u}}\,du = 4\sqrt{8}\,\sqrt{u}\Big]_0^2$$

$$= 4\sqrt{8}\sqrt{2}$$

$$= 4\sqrt{16} = 4\cdot 4 = 16.$$