Math 1272: Calculus II
Midterm II Review

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Find the tangent line to the parametric curve

$$x = t^2, \quad y = t\ln(t) - t$$

at $t = 2$.

At $t=2$, $\overset{x_0}{\overset{\shortparallel}{x}} = 2^2 = 4$, $y = 2\ln(2) - 2 = y_0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(t\ln(t) - t\right)}{\frac{d}{dt}\left(t^2\right)}$$

$$= \frac{\ln(t) + \cancel{\frac{t}{t} - 1}}{2t} = \frac{\ln(t)}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{\ln(2)}{4}$$

$$\boxed{y - y_0 = \frac{dy}{dx}(x - x_0)}$$

$x_0 = 4$

$y_0 = 2\ln(2) - 2$

$$\frac{dy}{dx} = \frac{\ln(2)}{4}$$

$$y - (2\ln(2) - 2) = \frac{\ln(2)}{4}(x - 4).$$

$$y = \frac{\ln(2)}{4}(x-4) + 2\ln(2) - 2$$

$$= \frac{\ln(2)}{4}x + \ln(2) - 2$$

$$\boxed{y = mx + b}$$

Find the solution of the differential equation

$$y' = \ln(x)y$$

separable

$$\frac{dy}{dx} = y' = \ln(x)y$$

$$\frac{1}{y}dy = \ln(x)\,dx$$

$$\int \frac{1}{y}dy = \int \ln(x)\,dx$$

I.B.P. $u = \ln(x)$

$dv = dx$

$$\ln|y| = x\ln(x) - x + C$$

$$|y| = e^{x\ln(x) - x + C}$$

$$= e^{C} e^{x\ln(x) - x}$$

$$\boxed{y = \pm e^{C} e^{x\ln(x) - x}}$$

$$y = A e^{x\ln(x) - x}$$

for any real number A

Solve the initial value problem

$$x^2y' + 2xy = \ln x, \quad y(1) = 2.$$

Linear : Integrating factor $I(x) = e^{\int P(x)\,dx}$

$$\boxed{y' + P(x)y(x) = Q(x)}$$

$$I(x)\left(y' + \underbrace{\frac{2}{x}}_{P} y\right) = \frac{\ln x}{x^2}\, I(x)$$

$$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$$

$$\underbrace{x^2y' + 2xy}_{} = \ln(x)$$

$$\frac{d}{dx}\left(I(x)y\right) = \ln(x)$$

$$\frac{d}{dx}(x^2y) = x^2y' + 2xy$$

$$\frac{d}{dx}\left(x^2 y\right) = \ln(x)$$

$$x^2 y = \int \ln(x)\,dx$$

$$x^2 y = x\ln(x) - x + C, \quad y(1) = 2$$

$$(1)^2 \cdot 2 = 1\cancel{\ln(1)} - 1 + C, \quad x=1,\ y=2$$

$$2 = -1 + C, \quad c = 3$$

Check

$$\boxed{y = \frac{\ln(x)}{x} - \frac{1}{x} + \frac{3}{x^2}}$$

Solve the initial value problem

$$xy' = y + x^2 \sin x, \quad y(\pi) = 0.$$

Linear: $\quad y' - \frac{1}{x} y = x \sin x \quad , \quad I(x) = e^{\int -\frac{1}{x}\,dx}$

$$= e^{-\ln(x)}$$

$$= e^{\ln\left(\frac{1}{x}\right)}$$

$$= \frac{1}{x}$$

$$\underbrace{\frac{1}{x}\left(y' - \frac{1}{x} y\right)} = \sin x$$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \sin x$$

$$\frac{1}{x} y' - \frac{1}{x^2} y$$

$$\frac{1}{x} y = \int \sin x \, dx = -\cos x + C$$

$$y = -x \cos x + Cx, \quad y(\pi) = 0$$

$$0 = y(\pi) = -\pi \cos(\pi) + C\pi$$

$$= \pi + C\pi, \quad C = -1$$

$$\boxed{y = -x \cos x - x}$$

A tank is filled with 100L of a salt-water mixture containing 10kg of dissolved salt. Brine containing 10g/L (1000 g = 1kg) of salt enters the tank at a rate of 1L/hour, while the salt-water solution in the tank drains at a rate of 2L/hour. Write down a differential equation for the amount of salt $S(t)$ in the tank for $t \leq 100$, assuming the salt-water mixture is always kept thoroughly mixed. How much salt is remaining after 10 hours?

$$\frac{dS}{dt} = \text{Flow in} - \text{Flow out} \quad \frac{kg}{hr}$$

$$\text{Flow in} = 10\,\frac{g}{L} \times 1\,\frac{L}{h} = 10\,\frac{g}{hr} = \frac{1}{100}\,\frac{kg}{hr}$$

$$\text{Amount of water in tank} = 100 - t \quad L$$

Concentration of salt $= \frac{S(t)}{100-t} \quad \frac{kg}{L}$

Flow out $= \frac{S(t)}{100-t} \frac{kg}{\cancel{L}} \times 2 \frac{\cancel{L}}{hr}$

$= \frac{2}{100-t} S(t) \quad \frac{kg}{hr}$

$$\boxed{\frac{dS}{dt} = \frac{1}{100} - \frac{2}{100-t} S(t)}$$

Linear

$$\frac{dS}{dt} + \underbrace{\left(\frac{2}{100-t}\right)}_{P(t)} S(t) = \frac{1}{100}$$

Integrating factor

$$I(t) = e^{\int P(t)\,dt}$$

$$= e^{\int \frac{2}{100-t}dt}$$

$$= e^{-2\ln(100-t)}$$

$$= e^{\ln\left(\frac{1}{(100-t)^2}\right)}$$

$$= \frac{1}{(100-t)^2}$$

$$I(t)\left(\frac{dS}{dt} + \frac{2}{100-t}S(t)\right) = \frac{1}{100}I(t)$$

$$\frac{d}{dt}\big(I(t)S(t)\big) = \frac{1}{100}I(t)$$

$$\int \frac{d}{dt}\left(\frac{S(t)}{(100-t)^2}\right)dt = \int \frac{1}{100(100-t)^2}\,dt$$

$$\frac{S(t)}{(100-t)^2} = \frac{1}{100(100-t)} + C$$

$$S(t) = \frac{100-t}{100} + c(100-t)^2$$

$$10 = S(0) = \frac{100}{100} + c(100)^2$$

$$= 1 + c\,100^2$$

$$c = \frac{9}{100^2}$$

$$S(t) = 1 - \frac{t}{100} + \frac{9}{100^2}(100-t)^2$$

$$S(t) = 1 - \frac{t}{100} + 9\left(1 - \frac{t}{100}\right)^2$$

After 10 hours

$$S(10) = 1 - \frac{10}{100} + 9\left(1 - \frac{10}{100}\right)^2$$

$$= 0.9 + 9(0.9)^2$$

$$\frac{d}{dt}\frac{1}{100-t} = -(100-t)^{-2}\cdot(-1)$$

$$= \frac{1}{(100-t)^2}$$

$$\int \frac{1}{(100-t)^2}\,dt = -\int \frac{1}{u^2}\,du$$

$u = 100-t$

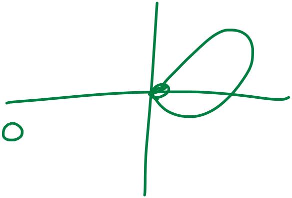
$du = -dt$

$$= \frac{1}{u} + C$$

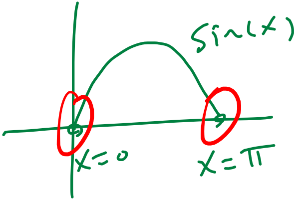
$$= \frac{1}{100-t} + C$$

Find the area of the region enclosed by one loop of the curve

$$r = 10\sin(\pi\theta).$$



Loop goes from $r=0 \longrightarrow r=0$



$$\sin(\pi\theta) : 0 \longrightarrow 0$$

$$\pi\theta : 0 \longrightarrow \pi$$

$$\theta : 0 \longrightarrow 1$$

$$\text{Area} = \int_0^1 \frac{1}{2} r^2 \, d\theta = \int_0^1 \frac{1}{2}\left(10\sin(\pi\theta)\right)^2 d\theta$$

$$= 50\int_0^1 \sin^2(\pi\theta)\,d\theta$$

$$= 50\int_0^1 \frac{1}{2} - \frac{1}{2}\cos(2\pi\theta)\,d\theta$$

$$= 25\int_0^1 1 - \cos(2\pi\theta)\,d\theta$$

u sub

$u = 2\pi\theta$

$du = 2\pi\,d\theta$

$$= 25\left(\theta - \cancel{\frac{\sin(2\pi\theta)}{2\pi}}\right)\Big|_0^1$$

$$= 25$$

small = 10 bees

A population of honeybees grows at a rate of 10 bees per hour when the population is small. The hive can contain at most 1000 bees. Use a logistic differential equation to model the honeybee population $P(t)$. If the population starts with 10 honeybee, how many bees will there be after 10 hours.

Logistic $\qquad \dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$

M = capacity = 1000 bees.

For small population $\qquad \dfrac{dP}{dt} \simeq kP$

$$10 \frac{\text{bees}}{\text{hr}} = k \cdot 10 \text{ bees}$$

$\Rightarrow k = 1$

$$10 \frac{\text{bees}}{\text{hr}} = k \cdot 10 \left(1 - \frac{10}{1000}\right)$$

$$1 = k\left(1 - \frac{1}{100}\right) = k \frac{99}{100}$$

$$\Rightarrow k = \frac{100}{99}$$

$$\frac{dP}{dt} = P\left(1 - \frac{P}{1000}\right), \quad \underset{P_0}{\overset{\shortparallel}{P(0)}} = 10$$

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad A = \frac{M - P_0}{P_0}$$

$$\boxed{P(t) = \frac{1000}{1+99e^{-t}}}$$

$$A = \frac{1000-10}{10}$$

$$= \frac{990}{10}$$

$$= 99$$

At $t = 10$ hours

$$P(10) = \frac{1000}{1+99e^{-10}}$$

Find dy/dx and d^2y/dx^2 for the parametric curve

$$x = t^3 + 1, y = t^2 - t.$$

For which values of t is the curve concave upward?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2 - t)}{\frac{d}{dt}(t^3+1)} = \frac{2t-1}{3t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2t-1}{3t^2}\right)}{3t^2}$$

$$= \frac{\dfrac{3t^2 \cdot 2 - (2t-1)\,6t}{(3t^2)^2}}{3t^2}$$

$$= \frac{6t^2 - 12t^2 + 6t}{(3t^2)^3}$$

$$= \frac{6t - 6t^2}{27t^6} = \frac{6}{27}\left(\frac{1-t}{t^5}\right)$$

$$\frac{d^2y}{dx^2} = \frac{2}{9}\left(\frac{1-t}{t^5}\right)$$

$1-t$ $+$ | $-$

$t=1$

t^5 $-$ | $+$

$t=0$

$\frac{d^2y}{dx^2}$ $-$ | $+$ | $-$

Concave down $t=0$ concave up $t=1$ concave down

Find the surface area obtained by rotating the curve

$$x = t^3, y = t^2, \quad 0 \leq t \leq 1$$

about the x-axis.

$$S = \int_{x(0)}^{x(1)} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx, \qquad x = x(t)$$

$$dx = x'(t)\, dt$$

$$= \int_0^1 2\pi y \sqrt{1 + \left(\frac{y'(t)}{x'(t)}\right)^2} \, x'(t)\, dt$$

$$\boxed{= \int_0^1 2\pi y(t) \sqrt{x'(t)^2 + y'(t)^2} \, dt}$$

$$x'(t) = 3t^2 \quad , \quad y'(t) = 2t$$

$$y(t) = t^2$$

$$SA = \int_0^1 2\pi t^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$= \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= \int_0^1 2\pi t^3 \sqrt{9t^2 + 4}\, dt$$

$$= \int_0^1 2\pi\, t\, t^2 \sqrt{9t^2+4}\, dt,$$

$$u = 9t^2 + 4$$

$$du = 18t\, dt$$

$$9t^2 = u - 4$$

$$t^2 = \frac{u-4}{9}$$

$$= \int_4^{13} 2\pi\left(\frac{u-4}{9}\right)\sqrt{u}\ \frac{du}{18}$$

$$= \frac{\pi}{81}\int_4^{13} \sqrt{u}\,(u-4)\ du$$

$$= \frac{\pi}{81}\int_4^{13} u^{3/2} - 4u^{1/2}\,du$$

$$= \frac{\pi}{81}\left(\frac{2}{5} u^{5/2} - 4 \cdot \frac{2}{3} u^{3/2} \right)_4^{13}$$

Find the arclength of the curve

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leq t \leq 1.$$

Find the arclength of the polar curve

$$r = 2(1 + \cos\theta).$$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4(1+\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4(1+2\cos\theta + \cos^2\theta) + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta$$

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