Math 1272: Calculus II
11.4 The comparison tests

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Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1} + \cdots$

Series converges if the sequence of partial sums

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

converges. ($\lim_{n\to\infty} S_n = S$). We write

$$S = \sum_{n=1}^{\infty} a_n$$ (Mostly $a_n > 0$ today).

We know the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. How about

$$\sum_{n=1}^{\infty} \frac{1}{n+1}, \quad \sum_{n=1}^{\infty} \frac{n}{n^2+n+1}, \quad \text{or} \; \sum_{n=2}^{\infty} \frac{1}{n-1}?$$

We know the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \qquad r = \frac{1}{2} < 1$$

converges. How about

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}, \quad \sum_{n=1}^{\infty} \frac{n}{2^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{2^n - 1}.$$

Comparison test

Suppose $\sum a_n$ and $\sum b_n$ are series with **positive** terms. Then

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

$$\sum a_n = a_1 + a_2 + a_3 + \cdots$$

$$\sum b_n = b_1 + b_2 + b_3 + \cdots$$

Note: The conditions $a_n \leq b_n$ or $a_n \geq b_n$ need only hold for $n \geq N$ for some fixed, possibly large N.

Generally compare against $\sum \frac{1}{n^p}$ or $\sum r^n$

$p > 1$ conv.

$|r| < 1$ conv.

Determine whether the series

$$\underbrace{\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}}$$

converges or diverges.

$$a_n \approx \frac{5}{2n^2} = \frac{5}{2} \frac{1}{n^2}$$

$$a_n = \frac{5}{2n^2 + \cancel{4n} + \cancel{3}} \leq \frac{5}{2n^2} = b_n \quad \text{for all } n \geq 1$$

$\sum b_n$ converges by p-test $(p = 2 > 1)$

$\Longrightarrow$ $\sum a_n$ converges by comparison test.

Example: Determine convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3+2n+3}$$

$$\frac{n+1}{n^3+2n+3} \approx \frac{n}{n^3} = \frac{1}{n^2} \quad \text{for large } n.$$

Compare against $\sum \frac{1}{n^2}$, which converges.

$$\underset{a_n}{\frac{n+1}{n^3+\cancel{2n}+\cancel{3}}} \le \frac{n+1}{n^3} \le \frac{n+n}{n^3} = \frac{2n}{n^3} = \frac{2}{n^2} = b_n$$

By comparison, series converges.
$p=2>1$

Test

$$\sum_{n=1}^{\infty} \underbrace{\frac{\ln n}{n}}_{a_n} \longrightarrow 0$$

for convergence/divergence.

If $\ln(n) \geq 1$ then $\dfrac{\ln(n)}{n} \geq \dfrac{1}{n}$

$\downarrow$

$n \geq e^1 \leq 3$

If $n \geq 3$ then $a_n = \dfrac{\ln(n)}{n} \geq \dfrac{1}{n} = b_n$

Since $\sum b_n$ diverges (Harmonic series)

$\sum a_n$ diverges by comparison.

Limit comparison test

Suppose $\sum a_n$ and $\sum b_n$ are series with **positive** terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a **finite** and **positive** number $(c > 0)$, then either both series converge or both diverge.

Ex: $\sum \frac{n+1}{n^3+2n+3}$ $\quad a_n = \frac{n+1}{n^3+2n+3}, \quad b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^3+2n+3}(n^2) = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+2n+3} \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n^2}+\frac{3}{n^3}} = \underbrace{1}_{=c} \rightarrow \text{converges.}$$

If $\quad b_n = \dfrac{1}{n^3}$, $\quad \dfrac{a_n}{b_n} = \dfrac{(n+1)(n^3)}{n^3+2n+3} \longrightarrow \infty$

$b_n = \dfrac{1}{n^{1.5}}$, $\quad \dfrac{a_n}{b_n} = \dfrac{(n+1)(n^{1.5})}{n^3+2n+3} \longrightarrow 0$

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{4n^2 - 2n}{\sqrt{1 + 2n^5}}$$

converges or diverges.

$$\frac{4n^2-2n}{\sqrt{1+2n^5}} \approx \frac{4n^2}{\sqrt{2n^5}} = \frac{4}{\sqrt{2}} \frac{n^2}{n^{2.5}}$$

$$= \frac{4}{\sqrt{2}} \frac{1}{\sqrt{n}}$$

Series looks like $\sum \frac{1}{n^p}$, $p=0.5$

Should diverge.

$p \le 1$ div.

$p > 1$ conv.

$$a_n = \frac{4n^2-2n}{\sqrt{1+2n^5}}, \quad b_n = \frac{1}{n^{0.5}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^2-2n}{\sqrt{1+2n^5}} \cdot n^{0.5}$$

$$= \lim_{n\to\infty} \frac{4n^{2.5}-2n^{1.5}}{\sqrt{1+2n^5}} \frac{\left(\frac{1}{n^{2.5}}\right)}{\left(\frac{1}{n^{2.5}}\right)}$$

$$= \lim_{n\to\infty} \frac{4-\cancel{\frac{2}{n}}^{\,0}}{\sqrt{\cancel{\frac{1}{n^5}}^{\,0}+2}} = \frac{4}{\sqrt{2}} > 0$$

Since $\sum \frac{1}{n^{0.5}}$ diverges (p-test)

$\sum a_n$ diverges.

Test $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+4}}{\underbrace{n^3+2n^2+4}_{b_n}} \approx \frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2}$

$a_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+2n^2+4}{n^2\sqrt{n^2+4}} \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{4}{n^3}}{\frac{1}{n}\sqrt{n^2+4}}$$

$\dfrac{1}{n} = \dfrac{1}{\sqrt{n^2}}$

$$= \lim_{n\to\infty} \frac{1+\cancel{\frac{2}{n}}+\cancel{\frac{4}{n^3}}}{\sqrt{1+\cancel{\frac{4}{n^2}}}} = 1$$

Since $p > 1$, converges.

By comparison

$$\frac{\sqrt{n^2+4}}{n^3+\cancel{2n^2}+\cancel{4}} \le \frac{\sqrt{n^2+4n^2}}{n^3} = \frac{\sqrt{5n^2}}{n^3}$$

$$\overset{\displaystyle a_n}{\shortparallel}$$

$$= \sqrt{5}\,\frac{n}{n^3} = \frac{\sqrt{5}}{n^2} = b_n$$

Converges by comparison.

Estimating sums

Use the sum of the first 100 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2}.$$

Estimate the error involved in the approximation.

Estimating sums

Use the sum of the first 100 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2}.$$

$$a_n = \frac{1}{n^2+2} \leq \frac{1}{n^2}$$

Estimate the error involved in the approximation. A computer program produces

$$\sum_{n=1}^{100} \frac{1}{n^2+2} = 0.8510785905657153.$$

Accuracy/Error? R_{100} = remainder

$$R_{100} = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{100} a_n = \sum_{n=101}^{\infty} a_n$$

$$= a_{101} + a_{102} + a_{103} + \cdots$$

$$R_{100} \leq \sum_{n=101}^{\infty} \frac{1}{n^2} \leq \int_{100}^{\infty} \frac{1}{x^2}\,dx$$

$$\lim_{T \to \infty}$$

$$= -\frac{1}{x}\Big]_{100}^{\infty}$$

$$= \frac{1}{100}$$