Math 1272: Calculus II
11.6 Ratio/Root tests

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Absolute convergence

A series $\sum a_n$ is **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Fact: An absolutely convergent series is convergent.

Examples

- $\sum \frac{(-1)^n}{n^2}$, $a_n = \frac{(-1)^n}{n^2}$, $|a_n| = \frac{1}{n^2}$, $\sum |a_n|$ converges
  $\longrightarrow$ absolute convergence

- $\sum \frac{(-1)^n}{n}$, $a_n = \frac{(-1)^n}{n}$, $|a_n| = \frac{1}{n}$, $\sum |a_n|$ diverges
  $\longrightarrow$ <u>not</u> absolutely conv.

Conditional convergence

A series $\sum a_n$ is **conditionally convergent** if it is convergent, but not absolutely convergent.

Example: The alternating series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Conditional convergence

A series $\sum a_n$ is **conditionally convergent** if it is convergent, but not absolutely convergent.

Fact: Any conditionally convergent series $\sum a_n$ can be rearranged and made convergent to **any** number s.

Example: Determine whether $\sum \frac{\cos(n)}{n^2}$ is convergent or divergent.

$a_n = \frac{\cos(n)}{n^2}$, $\quad |a_n| = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2} = b_n$

By comparison test $\sum |a_n|$ converges.

$\longrightarrow \sum a_n$ converges absolutely

Ratio test

Suppose $\sum a_n$ is a series for which the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists (possibly $L = \infty$). Then

- If $L < 1$ then the series $\sum a_n$ is **absolutely convergent**.
- If $L > 1$ then the series $\sum a_n$ is **divergent**.
- If $L = 1$ then the ratio test is **inconclusive**.

$\sum a_n$, $\qquad L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Assume $L < 1$. Then for any $L < r < 1$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| \le r \quad \text{for } n \ge N$$

(N depends on r).

$$\left|\frac{a_n}{a_{n-1}}\right| \le r \qquad (n-1 \ge N).$$

$$|a_n| \le r\,|a_{n-1}| \quad , \quad \left|\frac{a_{n-1}}{a_{n-2}}\right| \le r, \quad n-2 \ge N$$

$$\le r \cdot r\,|a_{n-2}| \qquad \left|\frac{a_{n-2}}{a_{n-3}}\right| \le r, \quad n-3 \ge N$$

$$\le r \cdot r \cdot r\,|a_{n-3}|$$

$$= r^3\,|a_{n-3}|$$

$$\vdots$$

$$= r^K\,|a_{n-K}| \qquad \text{provided } n-K \ge N$$

$$\text{Set } n-K = N$$

$$= r^{n-N}\,|a_N| \qquad K = n-N$$

$$= \left[\frac{|a_N|}{r^N}\right] r^n = A r^n$$

Then $|a_n| \le A r^n$, $0 < r < 1$

The geometric series $\sum A r^n$ converges $(r<1)$

So $\sum |a_n|$ converges by comparison test

$\longrightarrow$ $\sum a_n$ converges absolutely

If $L > 1$, $\left|\frac{a_{n+1}}{a_n}\right| > r$ $\quad (L > r > 1)$

$\leadsto$ $|a_n| \ge A r^n$ $\quad (r > 1)$

$\longrightarrow$ $\sum a_n$ diverges.

Example: Apply the ratio test to

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p > 0).$$

$$\left[\begin{array}{l} \text{conv. for } p > 1 \\ \text{div. for } p \le 1 \end{array}\right]$$

$$a_n = \frac{1}{n^p},$$

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\frac{1}{a_n} = n^p$$

$$= \lim_{n \to \infty} \left| \frac{n^p}{(n+1)^p} \right|$$

$$= \lim_{n \to \infty} \frac{n^p}{(n+1)^p} \quad \frac{\left(\frac{1}{n^p}\right)}{\left(\frac{1}{n^p}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{\left(1+\cancel{\frac{1}{n}}_{\,0}\right)^p} = 1$$

$\rightarrow$ Ratio test inconclusive

Example: Test the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$$

$$a_n = \frac{(-1)^n n^3}{3^n}$$

for absolute convergence.

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} , \quad \frac{1}{a_n} = \frac{3^n}{(-1)^n n^3}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{\cancel{3}^{n+1}}\cdot\frac{\cancel{3}^n}{n^3}$$

$$= \lim_{n\to\infty}\frac{1}{3}\frac{(n+1)^3}{n^3}$$

$$= \lim_{n\to\infty}\frac{1}{3}\left(1+\cancel{\frac{1}{n}}\right)^3 = \frac{1}{3} < 1$$

$$\Longrightarrow \sum \frac{(-1)^n n^3}{3^n} \text{ conv. absolutely.}$$

Example: Test the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

for absolute convergence.

$$a_n = \frac{n!}{n^n}$$

$$\frac{1}{a_n} = \frac{n^n}{n!}$$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{n^n}{\cancel{n!}} \cdot \frac{(n+1)\cancel{!}}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \cdot \frac{\left(\frac{1}{n^n}\right)}{\left(\frac{1}{n^n}\right)}$$

$$\frac{(n+1)!}{n!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}\,(n+1)}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1 \rightsquigarrow \text{absolute convergence}$$

$$b_n = \left(1+\frac{1}{n}\right)^n \qquad x = \frac{1}{n}$$

$$\ln(b_n) = n \ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \ln(b_n)$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

$$\lim_{n\to\infty} \ln(b_n) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

$$\longrightarrow \quad \lim_{n \to \infty} b_n = e^1 = e.$$

Root test

Suppose $\sum a_n$ is a series for which the limit

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$$

exists (possibly $L = \infty$). Then

- If $L < 1$ then the series $\sum a_n$ is **absolutely convergent**.
- If $L > 1$ then the series $\sum a_n$ is **divergent**.
- If $L = 1$ then the root test is **inconclusive**.

$L < 1$, $\quad L < r < 1$

$\sqrt[n]{|a_n|} < r \rightsquigarrow |a_n| < r^n$ (geometric series).

Example: Test the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+2} \right)^n$$

$$a_n = \left(\frac{n+1}{3n+2} \right)^n$$

for absolute convergence.

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n+1}{3n+2} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n \to \infty} \frac{1 + \cancel{\frac{1}{n}}^{0}}{3 + \cancel{\frac{2}{n}}^{0}} = \frac{1}{3} < 1$$

$\leadsto$ converges absolutely by <u>root test</u>

Example: Apply the root test to

$$\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2+\pi}\right)^n. \quad , a_n = \left(\frac{n^2}{n^2+\pi}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2}{n^2+\pi}\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} \qquad \boxed{\sqrt[n]{|a_n|} = \frac{n^2}{n^2+\pi}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\cancel{\frac{\pi}{n^2}}_{\,0}} = 1 = L.$$

$\rightsquigarrow$ Root test inconclusive.

$$a_n = \left(\frac{n^2}{n^2+\pi}\right)^n$$

$$= \left(\frac{1}{1+\frac{\pi}{n^2}}\right)^n$$

$$= \frac{1}{\left(1+\frac{\pi}{n^2}\right)^n} \longrightarrow \neq 0 \text{ as } n \to \infty$$

$\longrightarrow$ diverges.

$$m = \frac{n}{\pi}, \quad \frac{\pi}{n} = \frac{1}{m}$$

$$b_n = \left(1+\frac{\pi}{n^2}\right)^n \le \left(1+\frac{\pi}{n}\right)^n = \left(1+\frac{1}{m}\right)^{\pi m}$$

$$= \Big[\underbrace{\left(1+\frac{1}{m}\right)^m}_{\longrightarrow e \text{ as } m \to \infty}\Big]^{\pi}$$

$$\lim_{n\to\infty} b_n \le e^{\pi}$$

Example: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$, Since $\dfrac{1}{\ln(n)}$ is decreasing,

use alternating series test.

($\ln(n)$ is increasing).

Since $\displaystyle\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ and $\dfrac{1}{\ln(n)}$ decreasing

Converges by Alt. series test.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{n\,10^{n+1}}$ $\qquad (-9)^n = (-1)^n 9^n$

If we used alternating series test, need

$$\frac{9^n}{n\,10^{n+1}} \quad \text{decreasing.}$$

$$\frac{9^n}{n\,10^{n+1}} = \frac{9^n}{10n\,10^n} = \frac{1}{10n}\left(\frac{9}{10}\right)^n \quad \text{decreasing}$$

and $\displaystyle\lim_{n\to\infty} \underbrace{\frac{1}{10n}}_{\to 0}\underbrace{\left(\frac{9}{10}\right)^n}_{\to 0} = 0$

So converges by Alt series test.

Ratio test: $|a_n| = \dfrac{9^n}{n\,10^{n+1}}$ $\qquad |a_{n+1}| = \dfrac{9^{n+1}}{(n+1)\,10^{n+2}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{9^{n+1}}}{\cancel{(n+1)}\,10^{\cancel{n+2}}} \cdot \frac{\cancel{n\,10^{n+1}}}{\cancel{9^n}} = \left(\frac{n}{n+1}\right)\frac{9}{10}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{9}{10} < 1 \rightsquigarrow \text{convergence.}$$

Root test

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{10n}\left(\frac{9}{10}\right)^n} = \frac{9}{10}\sqrt[n]{\frac{1}{10n}}$$

≤ 1

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \frac{9}{10} < 1$$

Example:

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \simeq n^{3/2} \approx \frac{1}{n^{3/2}}$$

$$|a_n| = \left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right| \leq \frac{1}{\cancel{1}+n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

Since $\sum \frac{1}{n^{3/2}}$ converges $\left(p = 3/2 > 1\right)$

$\sum a_n$ converges absolutely by comparison.

Note: Limit comparison won't work.

Example: $\sum_{n=2}^{\infty} \left(\frac{n}{\ln(n)}\right)^n$ $\quad \left(\frac{n}{\ln(n)}\right)^n \geq 1 \rightarrow$ diverge.

Root test $\quad \sqrt[n]{|a_n|} = \frac{n}{\ln(n)}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n}{\ln(n)}$$

l'Hospital $\rightarrow$ $= \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty > 1$

$\longrightarrow$ diverges by root test

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n\,6^{2n}}{10^{n+1}}$

$$a_{n+1} = \frac{(n+1)\,6^{2(n+1)}}{10^{n+2}}$$

Ratio test:

$$\frac{1}{a_n} = \frac{10^{n+1}}{n\,6^{2n}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\,6^{\cancel{2n}+2}}{10^{\cancel{n+2}}} \cdot \frac{\cancel{10^{n+1}}}{n\,\cancel{6^{2n}}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{6^2}{10}$$

$$= \lim_{n\to\infty} \left(1+\underbrace{\cancel{\tfrac{1}{n}}}_{0}\right)\frac{36}{10} = \frac{18}{5} > 1$$

$\to$ diverges.

Roottest: $\sqrt[n]{|a_n|} = \left(\frac{n\,6^{2n}}{10^{n+1}}\right)^{\frac{1}{n}}$

$$= \frac{n^{\frac{1}{n}}\,6^{\frac{2n}{n}}}{10^{\frac{n+1}{n}}}$$

$$= \left(\frac{n}{10}\right)^{\frac{1}{n}} \frac{6^2}{10} > 1$$

$\downarrow$

1 as $n \to \infty$

$$b_n = \sqrt[n]{\frac{1}{10n}} = \left(\frac{1}{10n}\right)^{1/n}, \qquad x = \frac{1}{n}$$

$$n \to \infty$$

$$x \to 0$$

$$\ln(b_n) = \frac{1}{n}\ln\left(\frac{1}{10n}\right)$$

$$= x\ln\left(\frac{x}{10}\right) = x\ln(x) - \cancel{x\ln(10)} \quad 0 \quad x \to 0$$

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}}.$$

l'Hospital

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0} -x = 0.$$

$$\Rightarrow \lim_{n \to \infty} \ln(b_n) = 0$$

$$\Rightarrow \lim_{n \to \infty} b_n = e^0 = 1$$