Math 1272: Calculus II
11.8 Power series

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/1272S19

Computing special functions

Question: How does your calculator compute

$$\sin(x),\ \cos(x),\ \ln(x), \ldots ?$$

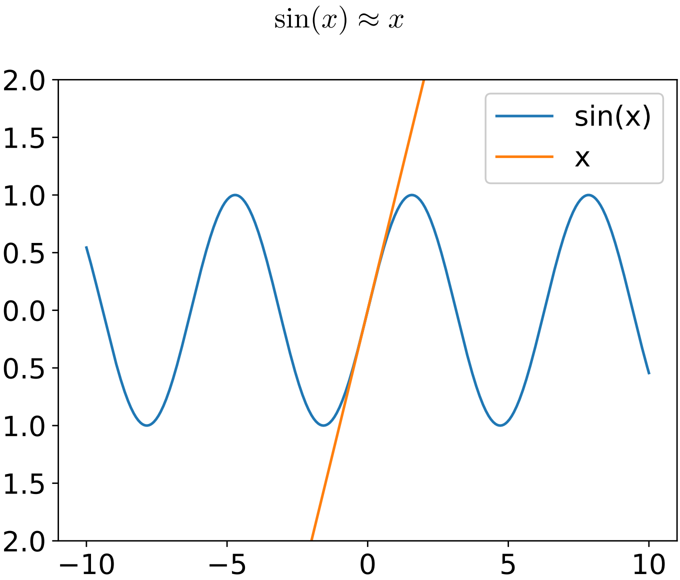
Type sin(1) into Google search:

$$\sin(1) = 0.8414709848\ldots$$

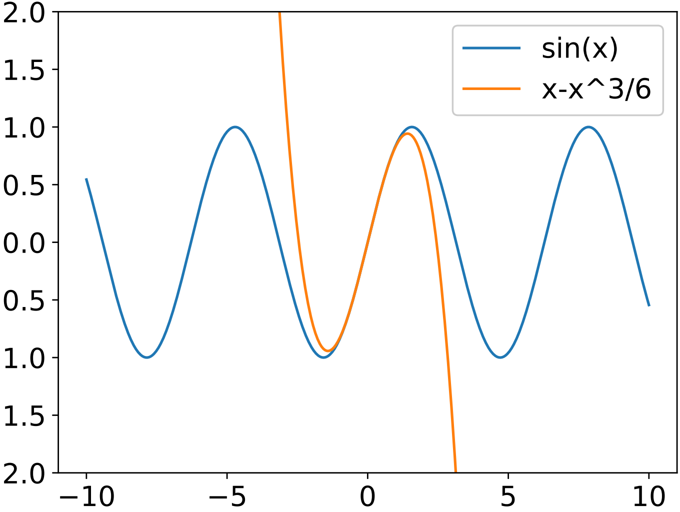
Approximating with simpler functions

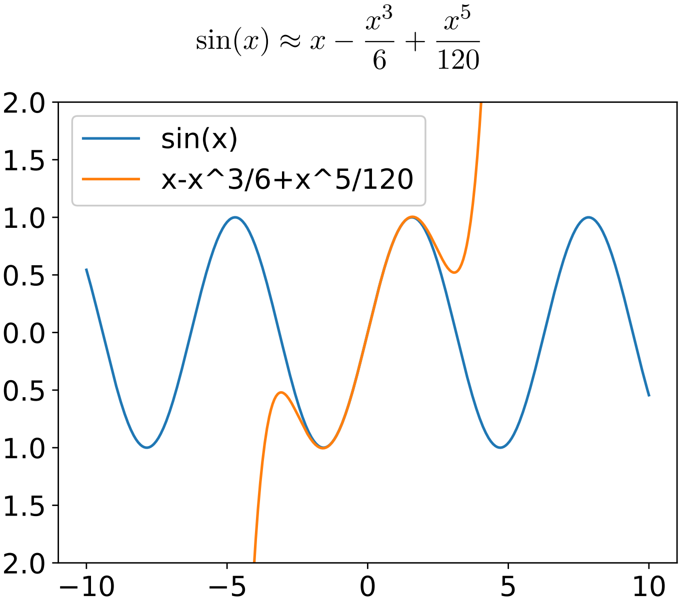
We can try to approximate a special function like $\sin(x)$, by simpler (computable) functions, such as polynomials:

$$\sin(x) \approx c_1 + c_2 x + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n.$$

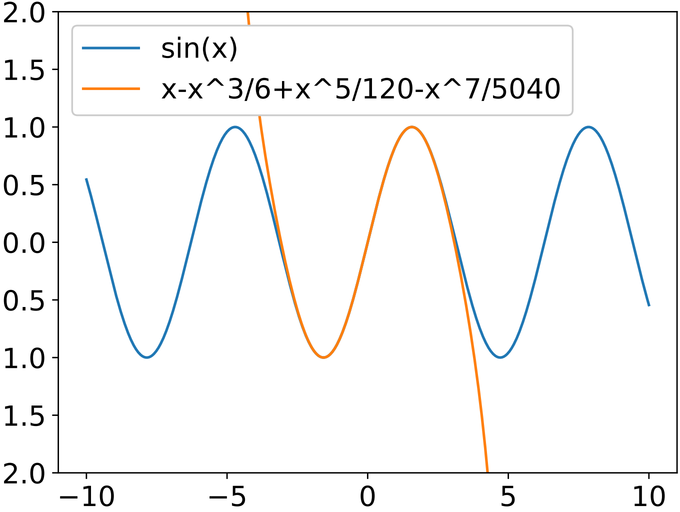

$\sin(x) \approx x$
sin(x)
x
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
−10
−5
0
5
10

$$\sin(x) \approx x - \frac{x^3}{6}$$

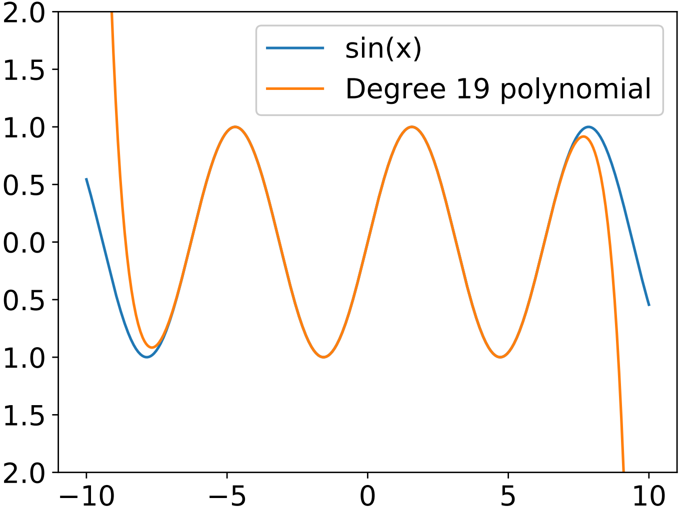



$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120}$
sin(x)
x-x^3/6+x^5/120
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
−10
−5
0
5
10

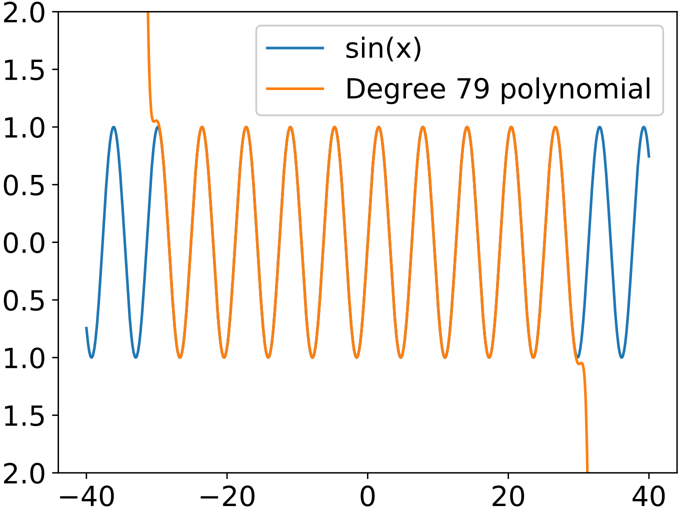
$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$



$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots + \frac{x^{17}}{355687428096000} - \frac{x^{19}}{121645100408832000}$$



$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots - \frac{x^{79}}{894618213078297528685144171539831652069808 2\ldots}$$



Questions:

- How did we come up with these polynomials?
- On what domain do they well-approximate $\sin(x)$?
- If we take the limit as degree $n \to \infty$, do the polynomials converge to $\sin(x)$ on the whole number line?

Power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots .$$

A power series may converge for some x and diverge for others.

Power series

A **power series** can also be translated

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

Example: If $c_n = 1$ we get the geometric series

$$\sum_{n=0}^{\infty} x^n.$$

Converges for $|x|<1$. to $\frac{1}{1-x}$

So

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } -1<x<1$$

diverges for $|x| \geq 1$

Example: For what values of x does the power series

$$\sum_{n=0}^{\infty} \frac{(x-1)^{\cancel{3}n}}{n^2}$$

converge?

<u>Ratio test</u>

$$|a_n| = \frac{|x-1|^n}{n^2}, \quad |a_{n+1}| = \frac{|x-1|^{n+1}}{(n+1)^2}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{|x-1|^{n+1}}}{(n+1)^2} \frac{n^2}{\cancel{|x-1|^n}}$$

$$= \lim_{n\to\infty} \underbrace{\cancel{\left(\frac{n}{n+1}\right)^2}}_{\to 1} |x-1| = |x-1|$$

$$L = |x-1|$$

Converges for $|x-1| < 1$
diverges for $|x-1| > 1$ } by ratio test.

Case 1 $x-1=1 \Rightarrow x=2$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges by p-test

Case 2 $x-1=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by AST

$\Rightarrow$ Converges for $\boxed{|x-1| \leq 1}$

Radius of convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there area only three possibilities:

1. The series converges only when $x = a$.

2. The series converges for all x.

3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x - a| > R$.

The number R is called the **radius of convergence**.

The series may converge or diverge for $x - a = R$ and $x - a = -R$. The possible **intervals of convergence** are

$$(a - R, a + R), (a - R, a + R], [a - R, a + R), [a - R, a + R].$$

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} x^n.$$

Converges for $|x| < 1$, $|x - 0| < 1 = R$

diverges for $|x| \geq 1$, $|x - 0| \geq 1 = R$

Radius of convergence $R = 1$

Interval of convergence $(-1, 1)$

$-1 < x < 1$

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-1)^{\cancel{3}n}}{n^2}$$

Found last class that

converges for $|x-1| \leq 1 = R$

diverges for $|x-1| > 1$

Radius of convergence $R = 1$

Interval of convergence $[-2, 0]$, $-2 \leq x \leq 0$

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Ratio test: $|a_{n+1}| = \dfrac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}}$, $\dfrac{1}{|a_n|} = \dfrac{\sqrt{n+1}}{3^n |x|^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} = 3|x| < 1$$

Converges for $|x| < \frac{1}{3}$

diverges for $|x| > \frac{1}{3}$

$\Big\}$ Radius of convergence $= \frac{1}{3}$

$\underline{x = -\frac{1}{3}}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

Compare to $\sum \frac{1}{n^{1/2}}$

$$\left[\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+n}} = \frac{1}{\sqrt{2}\cdot\sqrt{n}} \right]$$

Since $p = \frac{1}{2} < 1$, diverges by comp. test

$\underline{x = \frac{1}{3}}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

Converges by <u>alternating series test</u>.

Interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$

or $\quad -\frac{1}{3} < x \leq \frac{1}{3}$.

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

Ratio test:

$$|a_{n+1}| = \frac{(n+1)\,|x+2|^{n+1}}{3^{n+2}}, \quad \frac{1}{|a_n|} = \frac{3^{n+1}}{n\,|x+2|^n}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{|x+2|}{3}$$

$$= \lim_{n\to\infty} \left(1+\cancel{\tfrac{1}{n}}^{\,0}\right) \frac{|x+2|}{3} = \frac{|x+2|}{3} < 1$$

Converges for $|x+2| < 3 \Rightarrow (-5, 1)$

Div. for $|x+2|>3$

$x+2=3 \leadsto x=1$

$x+2=-3 \leadsto x=-5$

Check $x=1,-5$

$x=1$: $x+2=3$, $\sum_{n=0}^{\infty} \frac{n3^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{3}$

diverges

$x=-5$: $x+2=-3$, $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{3}$

diverges

Radius $=3$

Interval $(-5,1)$.