Math 1272: Calculus II
11.9 Representing functions via power series

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Power series as functions

We recall the **geometric series**

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n. \qquad |x| < 1$$

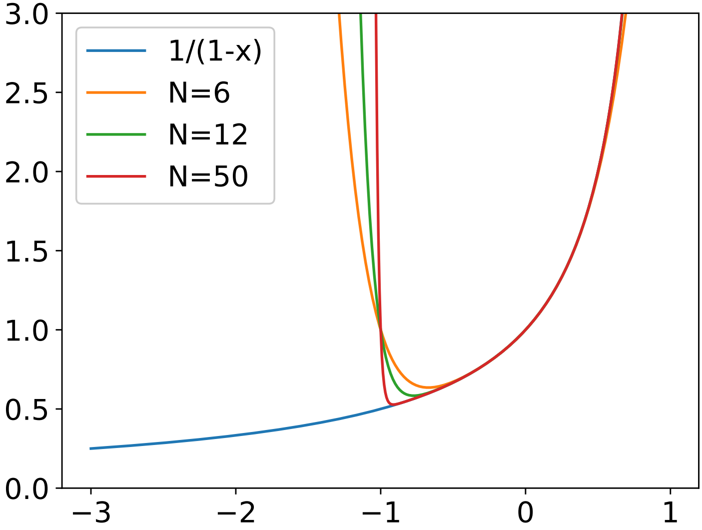
We treat the power series as a representation of the function

$$f(x) = \frac{1}{1-x}.$$

The partial sums approximate f, i.e.,

$$f(x) \approx \sum_{n=0}^{N} x^n$$

for large N.



1/(1-x)
N=6
N=12
N=50
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−3
−2
−1
0
1

Express $1/(1+x^3)$ as a sum of a power series and find the interval of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

Converges for $|-x^3| < 1 \rightarrow |x| < 1$

diverges for $|x| \geq 1$ $\qquad R = 1 =$ Radius

Interval of conv. $(-1, 1)$.

Find a power series representation for $1/(x+5)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{x+5} = \frac{1}{5} \cdot \frac{1}{1+\frac{x}{5}} = \frac{1}{5} \frac{1}{1-\left(-\frac{x}{5}\right)}$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{x}{5}\right)^n, \quad \left|-\frac{x}{5}\right| < 1$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}, \quad |x| < 5$$

Differentiation and Integration of Power Series

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Then

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx} c_n(x-a)^n = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$$

and

$$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

within the radius of convergence $|x-a| < R$.

Find a power series representation for $1/(1-x)^2$. What is the radius of convergence?

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{d}{dx}\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \frac{d}{dx}(1 + x + x^2 + \cdots)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} -n x^{n-1} = -(1 + 2x + 3x^2 + \cdots)$$

$$|x| < 1$$

$R = 1$ = Radius of convergence

Find a power series representation for $\ln(1+x)$. What is the radius of convergence?

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$$

$$\ln(1+x) = \int \frac{1}{1+x}\, dx$$

$$= \int \frac{1}{1-(-x)}\, dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\, dx \qquad |x| < 1.$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^n\, dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad |x| < 1$$

$$\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}, \qquad x = 1$$

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

↑ Conditionally convergent

Conditional convergence: By the last example we have

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots.$$

$$\frac{1}{2}\ln(2) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots$$

$$\frac{1}{2}\ln(2) = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + \cdots$$

$$+ \quad \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

$$\frac{3}{2}\ln(2) = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \cdots$$

$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots$$

$$? = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \cdots$$

$$= \ln(2)$$

$$\rightsquigarrow \frac{3}{2}\ln(2) = \ln(2)$$

$$3 = 2.$$

rearrangement not allowed.

Find a power series representation for ~~$\tan^{-1}(1+x)$~~. What is the radius of convergence?

$\tan^{-1}(x)$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n\,dx \rightsquigarrow |-x^2|<1$$

$$|x|<1$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad , \quad |x| < 1 = R.$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Evaluate

$$\int \frac{1}{1+x^7}\,dx$$

as a power series.

$$\int \frac{1}{1+x^7}\,dx = \int \frac{1}{1-(-x^7)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^7)^n \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{7n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$|-x^7| < 1$$

$$|x| < 1$$

$$-1 < x < 1$$