Math 1272: Calculus II
Midterm III Review

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Determine whether the series

$$\sum_{n=0}^{\infty} \frac{n^3 + 2n^2 + 3n - 2}{n^2 + 3n + 6} \simeq \frac{n^3}{n^2} = n$$

converges or diverges.

Recall if $\sum a_n$ converges the $\lim\limits_{n\to\infty} a_n = 0$

$\rightarrow$ diverges.

Determine the values of p for which the series

$$\sum_{n=0}^{\infty} \frac{\ln n}{n^p}$$

converges. Integral test

$p \neq 1$ $\displaystyle\int_1^{\infty} \frac{\ln x}{x^p}\,dx = \lim_{T\to\infty} \int_1^T \frac{\ln x}{x^p}\,dx$

$u = \ln x$, $du = \dfrac{dx}{x}$

$dv = \dfrac{1}{x^p}\,dx = x^{-p}\,dx$

$v = \dfrac{x^{-p+1}}{-p+1} = \dfrac{-1}{(p-1)x^{p-1}}$

$$= \lim_{T\to\infty} \left. \frac{-\ln x}{(p-1)x^{p-1}} \right]_1^T + \int_1^T \frac{1}{(p-1)x^p}\,dx$$

$$= \lim_{T\to\infty} \left[\frac{-\ln T}{(p-1)T^{p-1}} - \left. \frac{1}{(p-1)^2 x^{p-1}} \right|_1^T \right]$$

$$= \lim_{T\to\infty} \left(\frac{-\ln T}{(p-1)T^{p-1}} - \frac{1}{(p-1)^2 T^{p-1}} + \frac{1}{(p-1)^2} \right)$$

$p>1$ $$= \frac{1}{(p-1)^2}$$

Converges for $p>1$.

diverges for $p \le 1$ (compare to $\frac{1}{n^p}$).

Ex: Show $\frac{\ln(n)}{n^p}$ decreasing $n > N$.

Recall $\boxed{\begin{array}{l} a_n \ge 0 \\ b_n \ge 0 \end{array}}$ for limit comparison

Determine whether the series

$$\sum_{n=0}^{\infty} \underbrace{\frac{n^3 + 2n^2 + 3n - 2}{n^4 + 3n + 6}}_{a_n} \approx \frac{1}{n} = b_n$$

converges or diverges.

Comparison or limit comparison test.

$$\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \lim_{n\to\infty} \frac{n^3 + 2n^2 + 3n - 2}{n^4 + 3n + 6} \cdot n$$

$$= \lim_{n\to\infty} \frac{n^4 + 2n^3 + 3n^2 - 2n}{n^4 + 3n + 6} \frac{\left(\frac{1}{n^4}\right)}{\left(\frac{1}{n^4}\right)}$$

$$= \lim_{n\to\infty} \frac{1 + \cancel{\frac{2}{n}} + \cancel{\frac{3}{n^2}} - \cancel{\frac{2}{n^3}}^{0}}{1 + \cancel{\frac{3}{n^3}} + \cancel{\frac{6}{n^4}}} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by limit comparison by p-test $(p=1)$

the series diverges.

Determine whether the series

$$\sum_{n=0}^{\infty} \frac{n^3 + 2n^2 + 3n - 2}{n^4 + 3n + 6}$$

converges or diverges.

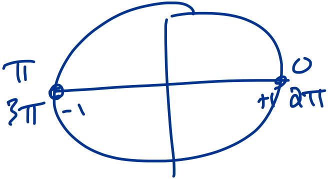
Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{\cos(\pi n)}{n}}$$

converges or diverges.

b_n must be decreasing for alt. series test.

$\cos(\pi n)$	n
$\cos(0) = 1$	0
$\cos(\pi) = -1$	1
$\cos(2\pi) = 1$	2
$\cos(3\pi) = -1$	3
$\cos(4\pi) = 1$	4



$$\cos(\pi n) = (-1)^n$$

diverges

$$\text{series} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

For what values of p does the series

$$\sum_{n=0}^{\infty}(-1)^n \underbrace{\frac{1}{n^p}}.$$

$b_n = \frac{1}{n^p}$

converge.

Alt. series test

① $\lim_{n\to\infty} b_n = 0$ and ② $b_n \geq b_{n+1} > 0$

$p > 0$ $\qquad$ $p \geq 0$

Converges for $p > 0$

converges absolutely for $p > 1$

Converges conditionally for $0 < p \leq 1$

Determine whether the series

$$\sum_{n=0}^{\infty}\left(\frac{101^n n^{102}}{n!}\right)^{\frac{1}{n}} = \frac{101\,(n^{102})^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}}$$

converges or diverges. Ratio test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{101^{n+1}(n+1)^{102}}{(n+1)!}\cdot\frac{n!}{101^n n^{102}}$$

$$= \lim_{n\to\infty} \frac{101}{n+1}\,\frac{(n+1)^{102}}{n^{102}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{101}{n+1}}_{\to 0}\Big(1+\underbrace{\tfrac{1}{n}}_{0}\Big)^{102} = 0 < 1$$

Therefore series converges absolutely by ratio test.

Find the radius of convergence and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \overbrace{\frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}}^{a_n}.$$

$|a_{n+1}|$ $\quad \frac{1}{|a_n|}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}} \left(\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots(2\cancel{n}-1)\right)}{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots(2\cancel{n}-1)(2(n+1)-1)\,\cancel{|x|^n}}$$

$$= \lim_{n\to\infty} \frac{|x|}{2(n+1)-1} = 0 < 1$$

By ratio test, converges for all x

Interval of convergence is $(-\infty,\infty)$, $R=\infty$

$$a_n = \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$a_1 = \frac{x}{1}, \quad n=1, \quad 2n-1=1$$

$$a_2 = \frac{x^2}{1 \cdot 3}, \quad n=2, \quad 2n-1=3$$

$$a_3 = \frac{x^3}{1 \cdot 3 \cdot 5}, \quad n=3, \quad 2n-1=5$$

$$a_{n+1} = a_n \cdot \frac{x}{2(n+1)-1}$$

Find the first 4 terms in the Maclaurin series for $f(x) = e^x \sin x$.

First 4 terms

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$\frac{f^{(n)}(0)}{n!}$$

$$f(0) = e^0 \sin(0) = 0$$

$$f'(x) = e^x \cos x + e^x \sin x$$

$$f'(0) = e^0 \cos(0) + e^0 \sin(0) = 1$$

$$f''(x) = e^x \cos x \cancel{- e^x \sin x} + \cancel{e^x \sin x} + e^x \cos x = 2e^x \cos x$$

$$f''(0) = 2$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x$$

$$f'''(0) = 2$$

$$0 + x + \frac{2}{2}x^2 + \frac{2}{6}x^3$$

$$= 0 + x + x^2 + \frac{1}{3}x^3 \approx e^x \sin x, \quad x \text{ near } 0$$

Taylor Series

Find the Maclaurin series for $f(x) = (1+x^2)e^x$

(1) Compute $f^{(n)}(0)$ for all n.

(2) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = (1+x^2)e^x = (1+x^2)\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \quad , \qquad \begin{array}{l} m = n+2 \\ n = m-2 \end{array}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{m=2}^{\infty} \frac{x^m}{(m-2)!}$$

$$= 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}$$

$$(1+x^2)e^x = 1 + x + \sum_{n=2}^{\infty} \left(\frac{1}{n!} + \frac{1}{(n-2)!} \right) x^n$$