Math 1272: Calculus II
7.8 Improper integrals

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/1272S19

Infinite intervals

If $\int_a^t f(x)\,dx$ exists for all $t > a$, then we define

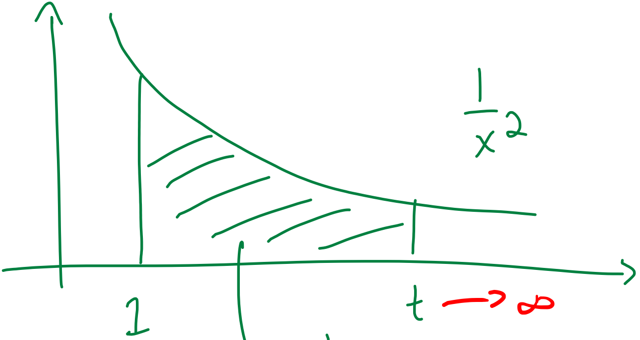
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists. The improper integral is called **convergent** if the corresponding limit exists and **divergent** otherwise.

Example 1. Compute $\int_1^\infty \frac{1}{x^2}\,dx$.

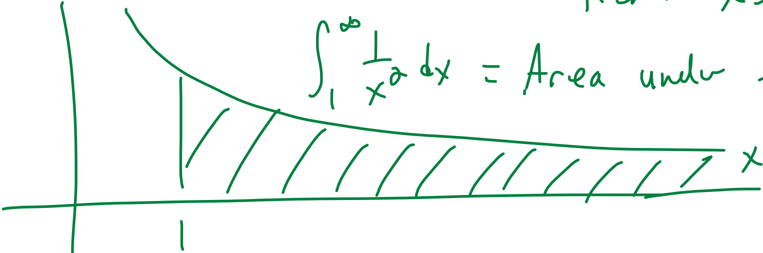
$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = -\frac{1}{t} - (-1) = 1 - \frac{1}{t}$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1 = \int_1^\infty \frac{1}{x^2}\,dx$$



$$\int_1^t \frac{1}{x^2}\,dx = \text{Area under } \frac{1}{x^2} \text{ from } x=1 \text{ to } x=t$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \text{Area under } \frac{1}{x^2} \text{ from } x=1 \text{ to } x=\infty$$



Infinite intervals

Example 2. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

$$\int \frac{1}{x^p}\,dx = \begin{cases} \ln x, & \text{if } p=1 \\ \dfrac{x^{-p+1}}{-p+1}, & \text{if } p \neq 1 \end{cases}$$

x^{-p}

If $p=1$, $\int_1^t \frac{1}{x}\,dx = \ln(t) - \ln(1) = \ln(t)$

Since $\lim_{t\to\infty} \ln(t) = \infty$, integral is divergent.

If $p \neq 1$, $\displaystyle\int_1^t \frac{1}{x^p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right]_1^t$

$$= \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

So

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \begin{cases} +\infty, & \text{if } p < 1 \\ -\dfrac{1}{-p+1}, & \text{if } p > 1 \end{cases}$$

$$\underbrace{-\frac{1}{-p+1}} = \frac{1}{p-1}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ is divergent if $p \le 1$

and convergent for $p > 1$

For $p > 1$, $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$

Infinite intervals

If $\int_t^b f(x)\,dx$ exists for all $t < b$, then we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists. The improper integral is called **convergent** if the corresponding limit exists and **divergent** otherwise.

If both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx.$$

We can take any real number a above.

Infinite intervals

Example 3. Compute $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

$$u = x^2, \quad du = 2x\,dx$$

$$x\,dx = \frac{1}{2}\,du$$

$$\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2}$$

$$\int_0^t xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big]_0^t = -\frac{1}{2}e^{-t^2} + \frac{1}{2}$$

So $$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \frac{1}{2}$$

$$\int_t^0 xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big]_t^0 = -\frac{1}{2} + \cancel{\frac{1}{2}e^{-t^2}}$$

$$\lim_{t \to -\infty} \int_t^0 x e^{-x^2}\, dx = -\frac{1}{2}$$

So $\int_{-\infty}^{\infty} x e^{-x^2}\, dx$ converges and

$$\int_{-\infty}^{\infty} x e^{-x^2}\, dx = \int_{-\infty}^{0} x e^{-x^2}\, dx + \int_{0}^{\infty} x e^{-x^2}\, dx$$

$$= -\frac{1}{2} + \frac{1}{2} = 0.$$

Singular integrands

If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

$t < b$

provided the limit exists. The improper integral is called **convergent** if the corresponding limit exists and **divergent** otherwise.

Example 4. Find $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$.

Singular integrands

If f is continuous on $(a, b]$ and discontinuous at a, then

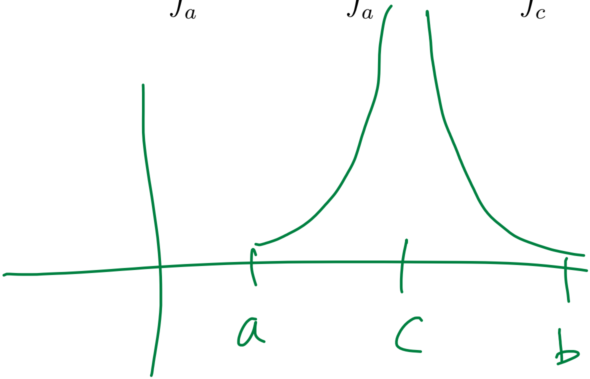
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided the limit exists. The improper integral is called **convergent** if the corresponding limit exists and **divergent** otherwise.

Singular integrands

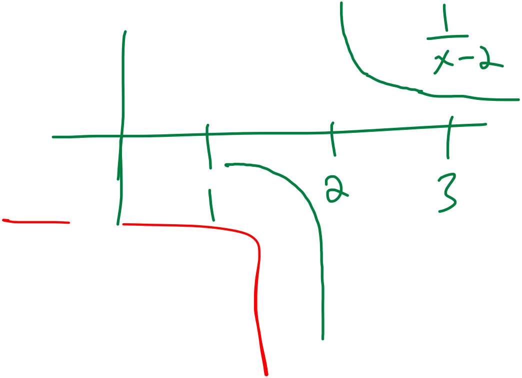
If f has a discontinuity at c where $a < c < b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



Singular integrands

Example 5. Compute $\int_0^3 \frac{1}{x-2}\,dx$.



Compute

$$\int_2^3 \frac{1}{\sqrt{x-2}}\,dx$$

$t>2$

$$\int_t^3 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{x-2}\Big]_t^3 = 2\sqrt{3-2} - 2\sqrt{t-2}$$

$$= 2 - 2\sqrt{t-2}$$

Take $\lim_{t \to 2^+} \int_t^3 \frac{1}{\sqrt{x-2}}\, dx = 2$

So $\int_2^3 \frac{1}{\sqrt{x-2}}\, dx$ is convergent

and $\int_2^3 \frac{1}{\sqrt{x-2}}\, dx = 2$

Comparison Theorem

Even if we cannot integrate a function exactly, we can still test for convergence or divergence.

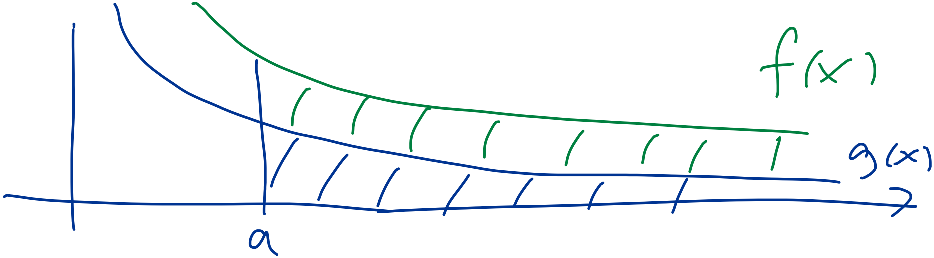
Theorem 1. *Suppose that f and g are continuous functions with*

$$f(x) \geq g(x) \geq 0 \quad \textit{for} \;\; x \geq a.$$

Then

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Comparison Theorem

Example 6. Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

We can integrate $\int_1^\infty e^{-x}\,dx$

$$\int_1^t e^{-x}\,dx = -e^{-x}\Big]_1^t = -e^{-t} + 1$$

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = 1$$

$\int_1^\infty e^{-x}\,dx$ converges.

Choose $f(x) = e^{-x}$. Notice

if $x \geq 1$ then $x^2 \geq x$, $(x \geq 1)$

Hence

$$-x^2 \leq -x \quad \text{if } x \geq 1$$

Since e^t is increasing in t,

$$\underset{g(x)}{\underset{\parallel}{e^{-x^2}}} \leq \underset{f(x)}{\underset{\parallel}{e^{-x}}} \quad \text{for } x \geq 1$$

$$\int_0^\infty e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{finite.}} + \int_1^\infty e^{-x^2}\,dx$$

Since $e^{-x^2} \le e^{-x}$ for $x \ge 1$ and

$\int_1^\infty e^{-x}\,dx$ converges, comparison theorem

yields $\int_1^\infty e^{-x^2}\,dx$ converges

and so does $\int_0^\infty e^{-x^2}\,dx$.