Math 1272 Section 40: Midterm I Solutions

Prof. Jeff Calder
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Name:________________________________ Section Number:________________________________

ID#:________________________________ Teaching Assistant:________________________________

Instructions:

1. *Please don't turn over this page until you are directed to begin.*
2. There are 5 problems on this exam, and problem 4 has two parts.
3. There are 8 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page you're working on, use the back of the page.
4. Please show all your work. Answers unsupported by an argument will get little credit.
5. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50

Some helpful formulas

$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$	$\sin^2 x + \cos^2 x = 1$	$2\sin^2 x = 1 - \cos(2x)$
$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$	$2\sin x \cos x = \sin(2x)$	$2\cos^2 x = 1 + \cos(2x)$
$\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert$	$\int \tan x \, dx = \ln\lvert\sec x\rvert$	$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right)$
$\int \csc x \, dx = \ln\lvert\csc x - \cot x\rvert$	$\int \cot x \, dx = \ln\lvert\sin x\rvert$	$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$	$\overline{x} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i$	$\overline{x} = \frac{1}{A}\int_a^b x f(x) \, dx$
$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx$	$\overline{y} = \frac{1}{m}\sum_{i=1}^{n} m_i y_i$	$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 \, dx$
$P = \rho g d$		

1. **(10 points)** Show that

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

[Hint: Use integration by parts.]

Solution. Integrate by parts, with $u = \sec\theta$, $du = \tan\theta\sec\theta\,d\theta$, $dv = \sec^2\theta\,d\theta$, $v = \tan\theta$, to find that

$$\begin{aligned}\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta.\end{aligned}$$

Adding $\int \sec^3\theta\,d\theta$ to both sides and dividing by 2 we have

$$\begin{aligned}\int \sec^3\theta\,d\theta &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.\end{aligned}$$ □

2. **(10 points)** Evaluate the **improper** integral

$$\int_8^{16} \frac{du}{u^2\sqrt{u^2-64}}.$$

Solution. Notice the function has a singularity at $u = 8$, so we compute

$$\int_8^{16} \frac{du}{u^2\sqrt{u^2-64}} = \lim_{t\to 8^+} \int_t^{16} \frac{du}{u^2\sqrt{u^2-64}}.$$

We use the trig substitution $u = 8\sec\theta$, so that

$$\sqrt{u^2-64} = \sqrt{64\sec^2\theta - 64} = 8\sqrt{\sec^2\theta - 1} = 8\tan\theta,$$

and $du = 8\tan\theta\sec\theta\, d\theta$. The limits are set by

$$t = 8\sec\theta \quad \text{and} \quad 16 = 8\sec\theta,$$

which gives $\cos\theta = \frac{8}{t}$ and $\cos\theta = \frac{1}{2}$, or $\theta = \cos^{-1}(\frac{8}{t})$ and $\theta = \cos^{-1}(\frac{1}{2})$. Thus we have

$$\begin{aligned}
\int_8^t \frac{du}{u^2\sqrt{u^2-64}} &= \int_{\cos^{-1}(\frac{8}{t})}^{\cos^{-1}(\frac{1}{2})} \frac{8\tan\theta\sec\theta}{64\sec^2\theta 8\tan\theta}\, d\theta \\
&= \frac{1}{64}\int_{\cos^{-1}(\frac{8}{t})}^{\cos^{-1}(\frac{1}{2})} \cos\theta\, d\theta \\
&= \frac{1}{64}\sin\theta\Big]_{\cos^{-1}(\frac{8}{t})}^{\cos^{-1}(\frac{1}{2})} \\
&= \frac{1}{64}[\sin(\cos^{-1}(\tfrac{1}{2})) - \sin(\cos^{-1}(\tfrac{8}{t}))] \\
&= \frac{\sqrt{3}}{128} - \frac{\sqrt{t^2-64}}{64t}.
\end{aligned}$$

Draw a triangle for the final computation. This gives

$$\int_8^{16} \frac{du}{u^2\sqrt{u^2-64}} = \frac{\sqrt{3}}{128} - \lim_{t\to 8^+} \frac{\sqrt{t^2-64}}{64t} = \frac{\sqrt{3}}{128}.$$

□

3. **(10 points)** Evaluate the integral

$$\int \frac{5x^2 - 2x + 3}{x^3 - x^2 + x - 1}\,dx.$$

Solution. We use partial fractions. The denominator factors as

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1),$$

where $x^2 + 1$ is irreducible. Thus, we look for a partial fraction expansion in the form

$$\frac{5x^2 - 2x + 3}{x^3 - x^2 + x - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}.$$

Cross multiplying we have

$$5x^2 - 2x + 3 = A(x^2+1) + (Bx + C)(x-1) = (A+B)x^2 + (C-B)x + A - C.$$

Hence $A + B = 5$, $C - B = -2$, and $A - C = 3$. Plugging $A = C + 3$ and $B = C + 2$ into $A + B = 5$ we have

$$C + 2 + C + 3 = 5 \implies C = 0.$$

Therefore $A = 3$, $B = 2$, and we have

$$\begin{aligned}\int \frac{5x^2 - 2x + 3}{x^3 - x^2 + x - 1}\,dx &= \int \frac{3}{x-1}\,dx + \int \frac{2x}{x^2+1}\,dx.\\ &= 3\ln|x-1| + \ln|x^2+1| + C.\end{aligned}$$

□

4. **(10 points total, 5 points each)**

(a) Find the area of the region $\mathcal{R}$ in the plane bounded by the curves

$$y = e^{-x}; \quad y = 0; \quad x = 0; \quad x = 1.$$

Solution. The area is

$$A = \int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 = -e^{-1} - (-e^0) = 1 - e^{-1}. \qquad \square$$

(b) Find the centroid (or center of mass) of the region $\mathcal{R}$.

Solution. The x-center of mass is

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^1 xe^{-x}\,dx \\
(IBP\ u = x, dv = e^{-x}\,dx) &= \frac{1}{A}(-xe^{-x})\Big]_0^1 - \frac{1}{A}\int_0^1 -e^{-x}\,dx \\
&= -\frac{e^{-1}}{A} - \frac{1}{A}\Big[e^{-x}\Big]_0^1 \\
&= \frac{-e^{-1} - e^{-1} + 1}{A} = \frac{1 - 2e^{-1}}{1 - e^{-1}}.
\end{aligned}$$

The y-center of mass is

$$\begin{aligned}
\overline{y} &= \frac{1}{A}\int_0^1 \frac{1}{2}[e^{-x}]^2\,dx \\
&= \frac{1}{A}\int_0^1 \frac{1}{2}e^{-2x}\,dx \\
&= -\frac{1}{4A}e^{-2x}\Big]_0^1 = \frac{1 - e^{-2}}{4(1 - e^{-1})}.
\end{aligned} \qquad \square$$

5. **(10 points)** Find the area of the surface obtained by rotating the curve

$$y = e^x + e^{-x}, \quad 0 \le x \le \ln(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})$$

about the x-axis. [Hint: Use substitutions $u = e^x - e^{-x}$ and $u = \tan\theta$, and use Problem 1.]

Solution. We have $\frac{dy}{dx} = e^x - e^{-x}$ and so the surface area is

$$S = 2\pi \int_0^{\ln(\frac{1}{2}+\frac{1}{2}\sqrt{5})} (e^x + e^{-x})\sqrt{1 + (e^x - e^{-x})^2}\, dx.$$

We make the substitution $u = e^x - e^{-x}$, $du = e^x + e^{-x}\, dx$, and so

$$S = 2\pi \int_0^1 \sqrt{1+u^2}\, du.$$

We make the second substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$ to obtain

$$S = 2\pi \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\sec^2\theta\, d\theta.$$

Using the identity $1 + \tan^2\theta = \sec^2\theta$ yields

$$S = 2\pi \int_0^{\pi/4} \sec^3\theta\, d\theta.$$

By Problem 1 we have

$$\begin{aligned} S &= \pi\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \\ &= \pi(\sqrt{2} + \ln(\sqrt{2}+1)) \\ &= \sqrt{2}\pi + \pi\ln(\sqrt{2}+1). \end{aligned}$$

□