Math 1272 Section 40: Midterm II

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Instructions:

1. *Please don't turn over this page until you are directed to begin.*
2. There are 5 problems on this exam.
3. There are 8 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page you're working on, use the back of the page.
4. Please show all your work. Answers unsupported by an argument will get little credit.
5. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50

Some helpful formulas

$\cos x \cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$	$\tan^2 x + 1 = \sec^2 x$	$1+\cot^2 x = \csc^2 x$
$\sin x \cos y = \frac{1}{2}[\sin(x-y)+\sin(x+y)]$	$\sin^2 x + \cos^2 x = 1$	$2\sin^2 x = 1-\cos(2x)$
$\sin x \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$	$2\sin x\cos x = \sin(2x)$	$2\cos^2 x = 1+\cos(2x)$
$\int \sec x\,dx = \ln\|\sec x + \tan x\|$	$\int \tan x\,dx = \ln\|\sec x\|$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$
$\int \csc x\,dx = \ln\|\csc x - \cot x\|$	$\int \cot x\,dx = \ln\|\sin x\|$	$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$L = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\overline{x} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i$	$\overline{x} = \frac{1}{A}\int_a^b x f(x)\,dx$
$S = \int_a^b 2\pi y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\overline{y} = \frac{1}{m}\sum_{i=1}^{n} m_i y_i$	$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$
$P(t) = \frac{M}{1+Ae^{-kt}},\quad A = \frac{M-P_0}{P_0}$	$I(x) = e^{\int P(x)\,dx}$	$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$
$x = r\cos\theta, y = r\sin\theta$	$A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$	$L = \int_a^b \sqrt{r^2+\left[\frac{dr}{d\theta}\right]^2}\,d\theta$
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$		

1. **(10 points)** Find the tangent line to the parametric curve

$$x = t^2 + t, \quad y = \sin(t)$$

at the point $(0, 0)$.

Solution. The slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos(t)}{2t+1}.$$

The point $(0, 0)$ corresponds to $t = 0$, and so $dy/dx = 1$. Hence the tangent line is $y = x$. $\square$

2. **(10 points)** Find the solution of the differential equation

$$y' = e^x e^{-y}$$

satisfying the initial condition $y(0) = \ln(2)$.

Solution. The equation is separable. We have

$$e^y dy = e^x dx,$$

and so

$$\int e^y dy = \int e^x dx.$$

Integrating both sides yields

$$e^y = e^x + C$$

for a constant C. Since $y(0) = \ln(2)$ we have

$$e^{\ln(2)} = e^0 + C$$

or

$$2 = 1 + C \implies C = 1.$$

Thus, the solution is

$$e^y = e^x + 1$$

or

$$y = \ln(e^x + 1).$$

$\square$

3. **(10 points)** Find the solution of the differential equation

$$xy' + y = \sqrt{x}$$

satisfying $y(1) = 2/3$.

Solution. We use the integrating factor method. The equation in standard form is

$$y' + \frac{1}{x}y = \frac{1}{\sqrt{x}}.$$

The integrating factor is

$$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|.$$

Due to the $\sqrt{x}$ in the equation, the domain of x is $x > 0$, so $I(x) = x$. Multiplying by $I(x)$ we have

$$xy' + y = \sqrt{x}$$

which is the form the equation originally came in. Now we know the left side is a product rule:

$$\frac{d}{dx}(xy) = \sqrt{x}.$$

Integrating we have

$$xy = \frac{2}{3}x^{3/2} + C.$$

Using $y(1) = 2/3$ we have

$$\frac{2}{3} = \frac{2}{3} + C.$$

Thus $C = 0$ and

$$y = \frac{2}{3}\sqrt{x}.$$

$\square$

4. **(10 points)** Find the area of the region enclosed by one loop of the curve

$$r = \cos(4\theta).$$

Solution. One loop goes between two zeros of $\cos(4\theta)$, so from $4\theta = -\pi/2$ to $4\theta = \pi/2$, or $\theta = -\pi/8$ to $\theta = \pi/8$. The area is

$$\begin{aligned}
\text{Area} &= \int_{-\pi/8}^{\pi/8} \frac{1}{2}\cos^2(4\theta)\,d\theta \\
\text{(Even integrand)} &= \int_{0}^{\pi/8} \cos^2(4\theta)\,d\theta \\
&= \int_{0}^{\pi/8} \frac{1}{2}(1+\cos(8\theta))\,d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{8}\sin(8\theta)\right)\Big]_0^{\pi/8} \\
&= \frac{\pi}{16}.
\end{aligned}$$

□

5. **(10 points)** A salt rock weighing 1kg is placed into a tank contaning 100L of water. The salt rock dissolves into the water in the tank at a rate of 1 gram per hour (1000 grams = 1 kg). You can assume the salt is always mixed well throughout the tank. The salt water mixture is leaking out of a hole in the bottom of the tank at a rate of 1L/hour. Write down a differential equation for the amount $S(t)$ of **dissolved** salt in the tank at time t. You do not need to solve the differential equation.

Solution. The flow into the tank (in kg of salt per hour) is the result of salt dissolving at a rate of 1/1000 kg per hour, so

$$\text{Flow in} = \frac{1}{1000}\text{kg/hour}.$$

The amount of water in the tank is changing since the water leaving the tank is not replaced. Since the tank starts with 100L and water is leaving at 1L/hour, the amount of water/salt mixture in the tank at t hours is $100 - t$. Hence, the concentration at time t is $S(t)/(100 - t)$, and so the flow out is

$$\text{Flow out} = \frac{S(t)\text{kg}}{(100-t)L} \times 1\text{L/hour} = \frac{S(t)}{100-t}\text{kg/hour}.$$

Hence, the differential equation is

$$\frac{dS}{dt} = \frac{1}{1000} - \frac{S(t)}{100-t}.$$

$\square$