1. (a) For \( k \in \mathbb{N} \) and \( \lambda > 0 \), consider the \((2k)\)th-order linear PDE
\[
(I - \lambda \Delta)^k u = f \quad \text{in } \mathbb{R}^n,
\]
where \( f \in L^2(\mathbb{R}^n) \). Use the Fourier Transform to formally derive the representation formula
\[
u = S_{k,\lambda} \ast f,
\]
where
\[
\hat{S}_{k,\lambda}(y) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1 + \lambda |y|^2)^k}.
\]
(b) Show, formally, that
\[
S_{k,\lambda}(x) = \frac{1}{(k-1)!} \frac{1}{(4\pi \lambda)^{n/2}} \int_0^\infty e^{-\frac{t-|x|^2}{4t\lambda}} t^{n/2-(k-1)} dt.
\]
[Hint: One way to do this is to first show that
\[
S_{k,\lambda} = S_{1,\lambda} \ast \cdots \ast S_{1,\lambda} = S_{k-1,\lambda} \ast S_{1,\lambda},
\]
and then use induction on \( k \).]

\textbf{Solution.} The hint is useful to see how to derive the expression for \( S_{k,\lambda} \). We can verify (4) more directly. Recall from class that
\[
\mathcal{F}(e^{-\sigma |x|^2}) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/4\sigma}.
\]
Let \( g(x) \) denote the right hand side of (4). Then setting \( \sigma = 1/4t\lambda \) we have
\[
\hat{g}(y) = \frac{1}{(k-1)!} \frac{1}{(4\pi \lambda)^{n/2}} \int_0^\infty e^{-\frac{t-|x|^2}{4t\lambda}} t^{n/2-(k-1)} dt
\]
\[
= \frac{1}{(k-1)!} \frac{1}{(2\pi)^{n/2}} \int_0^\infty t^{k-1} e^{-t(1+\lambda |y|^2)} dt.
\]
Make the substitution \( s = t(1+\lambda |y|^2) \) to obtain
\[
\hat{g}(y) = \frac{1}{(k-1)!} \frac{1}{(2\pi)^{n/2}} \frac{1}{(1+\lambda |y|^2)^k} \int_0^\infty s^{k-1} e^{-s} ds.
\]
Since
\[
\int_0^\infty s^{k-1} e^{-s} ds = \Gamma(k) = (k-1)!
\]
we have
\[
\hat{g}(y) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1+\lambda |y|^2)^k} = \hat{S}_{k,\lambda}(y).
\]
Therefore \( S_{k,\lambda} = g \). \( \square \)
(c) Fix \( \sigma > 0 \) and set \( \lambda(k) = \sigma^2/(2k) \). Show that
\[
S_{k,\lambda(k)} \to G_\sigma \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } k \to \infty,
\]
where
\[
G_\sigma(x) := \frac{1}{(2\sigma^2\pi)^{n/2}} e^{-\frac{|x|^2}{2\sigma^2}}.
\]

**Solution.** Let \( S_k = S_{k,\lambda(k)} \) and notice that
\[
\hat{S}_k(y) = \frac{1}{(2\pi)^{n/2}} \left( 1 + \frac{\sigma^2|y|^2}{2k} \right)^{-k}.
\]

We suppose that \( k \geq (n+1)/4 \) so that \( \hat{S}_k \in L^2(\mathbb{R}^n) \). Using the identity
\[
\left( 1 + \frac{1}{x} \right)^x < e < \left( 1 + \frac{1}{x} \right)^{x+1}
\]
with \( x = 2k/\sigma^2|y|^2 \) and \( y \neq 0 \) yields
\[
\hat{S}_k(y)g_k(y) \leq \frac{e^{-\sigma^2|y|^2/2}}{(2\pi)^{n/2}} \leq \hat{S}_k(y),
\]
where
\[
g_k(y) = \left( 1 + \frac{\sigma^2|y|^2}{2k} \right)^{-\sigma^2|y|^2/2}.
\]

The inequality above obviously holds when \( y = 0 \). Therefore
\[
\| \hat{S}_k - \hat{G}_\sigma \|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} |\hat{S}_k(y)|^2 |g_k(y) - 1|^2 \, dy.
\]

By the Dominated Convergence Theorem, the right hand side tends to 0 as \( k \to \infty \). Therefore
\[
\hat{S}_k \to \hat{G}_\sigma \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } k \to \infty.
\]

The result follows by an application of Plancherel’s Theorem
\[
\| \hat{S}_k - \hat{G}_\sigma \|_{L^2(\mathbb{R}^n)} = \| S_k - G_\sigma \|_{L^2(\mathbb{R}^n)}.
\]

It is worth thinking for a moment about the probabilistic interpretation of this limit (i.e, in the context of the central limit theorem).

2. Evans: Section 2.5, Problem 12 (Problem 10 in 1st edition)

3. Evans: Section 2.5, Problem 14 (Problem 12 in 1st edition)
Solution. You can use Duhamel’s principle or the Fourier transform method to derive the solution
\[
    u(x,t) = \int_0^t \int_{\mathbb{R}^n} e^{-c(t-s)} \Phi(x-y,t-s) f(y,s) \, dy \, ds + \int_{\mathbb{R}^n} e^{-ct} \Phi(x-y,t) g(y) \, dy,
\]
where \( \Phi(x,t) = e^{-|x|^2/4t} / (4\pi t)^{n/2} \) is the fundamental solution of the heat equation.

4. Evans: Section 2.5, Problem 15 (Problem 13 in 1st edition)

5. Give a direct proof that if \( U \) is bounded and \( u \in C^2(U_T) \cap C(\overline{U}_T) \) solves the heat equation, then
\[
    \max_{\overline{U}_T} u = \max_{\Gamma_T} u.
\]
[Hint: Define \( u_\varepsilon := u - \varepsilon t \) for \( \varepsilon > 0 \), and show that \( u_\varepsilon \) cannot attain its maximum over \( \overline{U}_T \) at a point in \( U_T \).]

6. Evans: Section 2.5, Problem 17 (Problem 14 in 1st edition)