Math 222A – Homework 4 (Due Oct 7)

1. Define
$$u(x,t) := \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \qquad (x,t) \in \mathbb{R} \times [0,\infty),$$
where
$$g(t) := \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$
Show that u is a solution of the heat equation
$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

2. **Comparison principle:** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u, v \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy
$$\begin{cases} u_t - \Delta u \le f & \text{in } \Omega_T \\ u \le g & \text{on } \Gamma_T, \end{cases}$$
and
$$\begin{cases} v_t - \Delta v \ge f & \text{in } \Omega_T \\ v \ge g & \text{on } \Gamma_T. \end{cases}$$
Show that $u \le v$ on $\overline{\Omega_T}$. [Remark: We call u a subsolution, and v a supersolution of the heat equation.]

3. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in C_1^2(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ be a solution of the heat equation
$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Omega \times \{t = 0\} \\ u = 0 & \text{on } \partial\Omega \times \{t > 0\}, \end{cases}$$
and let $u_\infty \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of
$$\begin{cases} -\Delta u_\infty = f & \text{in } \Omega \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$
Show that
$$\lim_{t\to\infty} u(x,t) = u_\infty(x) \quad \text{uniformly in } x.$$
[Hint: Use the comparison principle to compare u against super and subsolutions of the form
$$v(x,t) = u_\infty(x) \pm \varphi(x,t),$$
where $\lim_{t\to\infty} \varphi(x,t) = 0$ uniformly in x.]

4. Evans: Section 2.5, Problem 19 (Problem 15 in 1st Edition)

5. Evans: Section 2.5, Problem 24 (Problem 17 in 1st Edition)