Math 222A – Homework 4 Solutions

1. Define
$$u(x,t) := \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \qquad (x,t) \in \mathbb{R} \times [0,\infty),$$
where
$$g(t) := \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$
Show that u is a solution of the heat equation
$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

2. **Comparison principle:** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u, v \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy
$$\begin{cases} u_t - \Delta u \le f & \text{in } \Omega_T \\ u \le g & \text{on } \Gamma_T, \end{cases}$$
and
$$\begin{cases} v_t - \Delta v \ge f & \text{in } \Omega_T \\ u \ge g & \text{on } \Gamma_T. \end{cases}$$
Show that $u \le v$ on $\overline{\Omega_T}$. [Remark: We call u a subsolution, and v a supersolution of the heat equation.]

3. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in C_1^2(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ be a solution of the heat equation
$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Omega \times \{t = 0\} \\ u = 0 & \text{on } \partial\Omega \times \{t > 0\}, \end{cases}$$
and let $u_\infty \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of
$$\begin{cases} -\Delta u_\infty = f & \text{in } \Omega \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$
Show that
$$\lim_{t\to\infty} u(x,t) = u_\infty(x) \quad \text{uniformly in } x.$$
[Hint: Use the comparison principle to compare u against super and subsolutions of the form
$$v(x,t) = u_\infty(x) \pm \varphi(x,t),$$
where $\lim_{t\to\infty} \varphi(x,t) = 0$ uniformly in x.]

Solution. Since Ω is bounded, there exists $R > 0$ such that $\Omega \subseteq B(0, R-1)$. Let ξ be a smooth cutoff function satisfying $\xi \equiv 1$ in $B(0, R-1)$, $\xi \equiv 0$ in $\mathbb{R}^n \setminus B(0, R)$ and $0 \leq \xi \leq 1$. Define

$$\varphi(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)\xi(y)\, dy,$$

where Φ is the fundamental solution of the heat equation. Then φ satisfies the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n$, $\varphi \geq 0$ on $\mathbb{R}^n$, and $\varphi(x, 0) \equiv 1$ in Ω. Furthermore, we have

$$|\varphi(x,t)| = \frac{1}{(4\pi t)^{n/2}} \int_{B(0,R)} e^{-|x-y|^2/4t}\xi(y)\, dy \leq \frac{\alpha(n)R^n}{(4\pi t)^{n/2}}$$

for all $t > 0$ and $x \in \mathbb{R}^n$. Therefore

$$\lim_{t\to\infty} \varphi(x,t) = 0 \quad \text{uniformly in } x.$$

Let $C := \|u_\infty\|_{L^\infty(\Omega)}$ and set $v(x,t) := u_\infty(x) + C\varphi(x,t)$. Then v is a solution of the heat equation $v_t - \Delta v = f$ on $\Omega \times (0, \infty)$ that satisfies

$$v(x,t) = u_\infty(x) + C\varphi(x,t) \geq 0 \quad \text{for } x \in \partial\Omega,$$

and

$$v(x, 0) = u_\infty(x) + C \geq 0 \quad \text{for } x \in \Omega.$$

By the comparison principle, $u \leq u_\infty + C\varphi$ on $\Omega \times (0, \infty)$. A similar argument shows that $u \geq u_\infty - C\varphi$. Therefore

$$|u - u_\infty| \leq C\varphi \quad \text{ on } \Omega \times (0, \infty).$$

It follows that $\lim_{t\to\infty} u(x,t) = u_\infty(x)$ uniformly in x. Furthermore, we have the decay estimate

$$\|u(\cdot, t) - u_\infty(\cdot)\|_{L^\infty(\Omega)} \leq \frac{C}{t^{n/2}}.$$

□

4. Evans: Section 2.5, Problem 19 (Problem 15 in 1st Edition)

5. Evans: Section 2.5, Problem 24 (Problem 17 in 1st Edition)