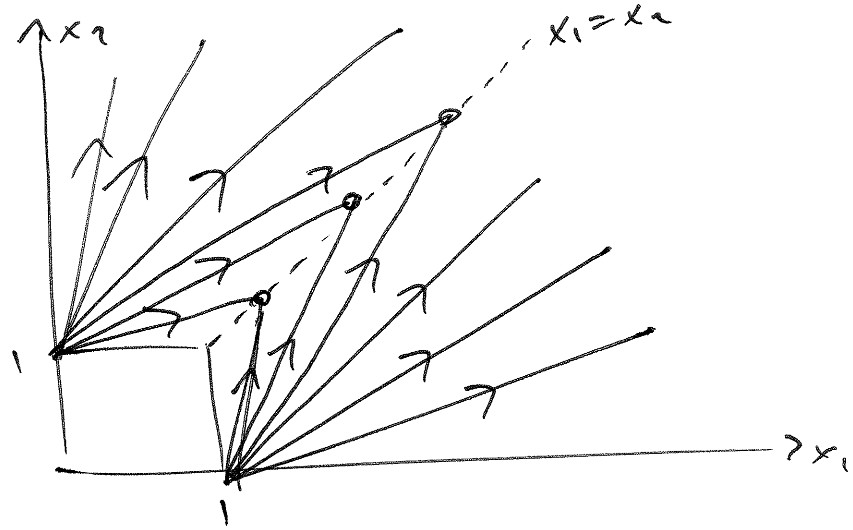
① Solution for 1e) $u(x) = \begin{cases} 2\sqrt{x_1(x_2-1)}, & x_2 \geq x_1 \\ 2\sqrt{x_2(x_1-1)}, & x_2 \leq x_1 \end{cases}$

Characteristics

Meet on line

$x_1 = x_2$



③ Let $w(x,t) = u(x,t) - v(x,t) - \varepsilon t$, $\varepsilon > 0$.

At a max $(x_0, t_0) \in U_T$ of w,

$w_t \geq 0$ and $Dw = 0$

$\Updownarrow \qquad\qquad \Updownarrow$

$u_t - v_t \geq \varepsilon \qquad Du = Dv$ at (x_0, t_0)

Subtract the PDE for u and v to find

$$u_t + H(Du, x) - \big(v_t + H(Dv, x)\big) \leq 0$$

or

$$u_t - v_t \leq H(Dv, x) - H(Du, x) = 0 \quad \text{at } (x_0, t_0)$$

This contradicts $u_t - v_t \geq \varepsilon > 0$.

Hence the maximum of w is attained on the boundary Γ_T, where $w \leq 0$
(Assume U bounded)

Hence $w \leq 0$ on $\overline{U_T}$, hence

$\forall \varepsilon > 0$, $u \leq v + \varepsilon t$ on $\overline{U_T}$

Sending $\varepsilon \to 0^+$ we have $u \leq v$ on $\overline{U_T}$.

If u and v are two solutions of

$$\begin{cases} u_t + H(Du, x) = 0, & \text{in } U_T \\ u = g, & \text{on } \Gamma_T \end{cases}$$

then previous argument shows that $u \leq v$ and $v \leq u$ in U_T. Hence $u = v$ in U_T.

$\blacksquare$

(5) Assume $v \in \partial H(p)$. Then

$$H(r) \geq H(p) + v\cdot(r-p) \quad \text{for all } r \in \mathbb{R}^n.$$

Hence
$$L(v) = \underset{r}{\overset{\text{sup}}{\cancel{\max}}} \{r\cdot v - H(r)\}$$

$$\leq \sup_r \{r\cdot v - H(p) - v\cdot(r-p)\}$$

$$= v\cdot p - H(p)$$

On the other hand, setting $r=p$ yields

$$L(v) \geq v\cdot p - H(p)$$

Therefore
$$H(p) + \overset{L}{\cancel{H}}(v) = v\cdot p \quad //$$

Assume
$$v\cdot p = H(p) + L(v). \quad \text{Then}$$

$$L(v) = \sup_r \{r\cdot v - H(r)\} \geq p\cdot v - H(p) = L(v).$$

Therefore
$$r\cdot v - H(r) \leq p\cdot v - H(p) \quad \forall r \in \mathbb{R}^n$$

$$\iff H(r) \geq H(p) + v\cdot(r-p) \quad \forall r \in \mathbb{R}^n$$

$$\iff v \in \partial H(p).$$

$\boxed{///}$

(b) We first show the min exists.

Since $\lim_{p\to\infty} \frac{L(p)}{|p|} = +\infty$, $\exists M>0$ such that

$$L(p) \geq (Lip(g)+1)|p| \quad \text{whenever} \quad |p| \geq M$$

let $y \in \mathbb{R}^n$ such that $tL\left(\frac{x-y}{t}\right) + g(y) \leq u(x,t) + 1$

If $\left|\frac{x-y}{t}\right| \geq M$ then

$$u(x,t) + 1 \geq (Lip(g)+1)|x-y| + g(y)$$

$$\geq (Lip(g)+1)|x-y| + g(x) - Lip(g)|x-y|$$

$$= |x-y| + g(x)$$

Hence $|x-y| \leq u(x,t) + g(x) + 1$

It follows that

$$|x-y| \leq \max\{u(x,t) + g(x) + 1,\ Mt\} =: C$$

whenever $tL\left(\frac{x-y}{t}\right) + g(y) \leq u(x,t) + 1$

Therefore

$$u(x,t) = \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

$$= \inf_{|x-y|\le C} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

$$= \min_{|x-y|\le C} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

$$= \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Now, at a minimum $y\in\mathbb{R}^n$, since L and g are C^1 we have

$$-DL\left(\frac{x-y}{t}\right) + Dg(y) = 0$$

or $$Dg(y) = DL\left(\frac{x-y}{t}\right)$$

By problem (5), $\frac{x-y}{t} = DH(Dg(y))$

Hence $$\left|\frac{x-y}{t}\right| \le R \overset{\text{def}}{=} \sup_{y\in\mathbb{R}^n} |DH(Dg(y))|$$ $\boxtimes$

⑧ Let $t_0 > 0$ and set $T = t_0 + 1$

Let $M > 0$ s.t. $u \equiv 0$ in

$$\mathbb{R} \times [0,T] \setminus \left([-M,M] \times [0,T] \right)$$

Let $V^n(x,t) = \varphi(x)\psi^n(t)$ where

$$\begin{cases} \varphi \equiv 1 & \text{on } [-M,M] \text{ and} \\ \varphi \equiv 0 & \text{on } \mathbb{R} \setminus [-M-1, M+1] \end{cases}$$

and $\psi^n(t) = \begin{cases} 1, & t \le t_0 \\ n(t_0 - t) + 1, & t_0 \le t \le t_0 + \frac{1}{n} \\ 0, & t \ge t_0 + \frac{1}{n}. \end{cases}$

Since u is an integral solution of

$$\begin{cases} u_t + F(u)_x = 0, & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

We have

$$0 = \int_0^\infty \int_{-\infty}^\infty u v_t^n + F(u) v_x^n \, dx\, dt + \int_{-\infty}^\infty g(x) v^n(x,0)\, dx \qquad (*)$$

[Aside: Technically $v^n \notin C_c^\infty$, so we should first mollify v^n: $v_\varepsilon^n \stackrel{\text{def}}{=} \eta^\varepsilon * v^n$. Write down $(*)$ for v_ε^n and send $\varepsilon \to 0^+$.]

Since $v_x^n = 0$ on the support of u, $(*)$ yields

$$\int_{-\infty}^\infty g(x)\, dx = \frac{1}{n} \int_{t_0}^{t_0 + \frac{1}{n}} \int_{-\infty}^\infty u(x,t)\, dx\, dt$$

$$\xrightarrow[\infty]{n} \int_{-\infty}^\infty u(x,t)\, dx$$

$\boxed{\text{Yu}}$