MATH 222A – HOMEWORK 8 (DUE MONDAY NOVEMBER 23)

1. Verify that if $n > 1$, the unbounded function $u = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U)$, for $U = B(0,1)$.

2. (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show that

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u)u_{x_i} \ (i = 1, \ldots, n).$$

3. Assume $1 \leq p \leq \infty$ and U is bounded.

   (a) Prove that if $u \in W^{1,p}(U)$ then $|u| \in W^{1,p}(U)$.

   (b) Prove that $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\}, \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

   Here, $u^+ = \max(0, u)$ and $u^- = -\min(0, u)$, so that $u = u^+ - u^-$. [Hint: $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, for

$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \geq 0 \\ 0 & \text{if } z < 0.] \end{cases}$$

   (c) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$