Math 222A – Homework 9 (Due Friday December 11)

1. Consider Laplace's equation with potential function c:

$$-\Delta u + cu = 0, \tag{1}$$

and the divergence structure equation:

$$-\operatorname{div}(aDv) = 0, \tag{2}$$

where the function a is positive.

   (a) Show that if u solves (1) and $w > 0$ also solves (1), then $v := u/w$ solves (2) for $a := w^2$.

   (b) Conversely, show that if v solves (2), then $u := va^{1/2}$ solves (1) for some potential c.

2. Evans: Section 6.6, Problem 2 (Problem 1 in 1st edition)

3. Evans: Section 6.6, Problem 3 (Problem 2 in 1st edition)

4. Evans: Section 6.6, Problem 4 (Problem 3 in 1st edition)

5. Explain how to define $u \in H^1(U)$ to be a weak solution of Poisson's equation with *Robin boundary conditions:*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases}$$

Discuss the existence and uniqueness of a weak solution for a given $f \in L^2(U)$.

6. Suppose U is connected and ∂U consists of two disjoint, closed sets Γ_1 and Γ_2. Define what it means for u to be a weak solution of Poisson's equation with *mixed Dirichlet-Neumann boundary conditions:*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0. & \text{on } \Gamma_2 \end{cases}$$

Discuss the existence and uniqueness of weak solutions.