Math 5467 – Homework 3 Solutions

1. (Split-Radix FFT) Assume $n \geq 4$ is a power of 2 and let $f \in L^2(\mathbb{Z}_n)$. Define $f_e \in L^2(\mathbb{Z}_{\frac{n}{2}})$, and $f_{o,1}, f_{o,2} \in L^2(\mathbb{Z}_{\frac{n}{4}})$ by

$$f_e(k) = f(2k), \quad f_{o,1}(k) = f(4k+1), \quad \text{and} \quad f_{o,2}(k) = f(4k+3).$$

   (i) Show that

$$\mathcal{D}_n f(\ell) = \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell). \tag{1}$$

Proof by Michael Markiewicz. By definition of the discrete Fourier transform, we have

$$\begin{aligned}
\mathcal{D}_n f(\ell) &= \sum_{k=0}^{n-1} f(k)e^{-2\pi ik\ell/n} \\
&= \sum_{k=0}^{\frac{n}{2}-1} f(2k)e^{-2\pi i(2k)\ell/n} + \sum_{k=0}^{\frac{n}{2}-1} f(2k+1)e^{-2\pi i(2k+1)\ell/n} \\
&= \sum_{k=0}^{\frac{n}{2}-1} f(2k)e^{-2\pi ik\ell/\frac{n}{2}} + e^{-2\pi i\ell/n}\sum_{k=0}^{\frac{n}{2}-1} f(2k+1)e^{-2\pi ik\ell/\frac{n}{2}} \\
&= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\left(\sum_{k=0}^{\frac{n}{4}-1} f(4k+1)e^{-2\pi i(2k)\ell/\frac{n}{2}} + \sum_{k=0}^{\frac{n}{4}-1} f(4k+3)e^{-2\pi i(2k+1)\ell/\frac{n}{2}}\right) \\
&= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\left(\sum_{k=0}^{\frac{n}{4}-1} f(4k+1)e^{-2\pi ik\ell/\frac{n}{4}} + e^{-2\pi i\ell/\frac{n}{2}}\sum_{k=0}^{\frac{n}{4}-1} f(4k+3)e^{-2\pi ik\ell/\frac{n}{4}}\right) \\
&= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{(-2\pi i\ell)(1/n+2/n)}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \\
&= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)
\end{aligned}$$

as desired. □

   (ii) The FFT algorithm based on the 3-way split in (1) is called the split-radix FFT algorithm. There are a lot of redundant computations in (1), and these must be accounted for in order to realize the improved complexity of the split-radix FFT. Show that

$$\begin{aligned}
\mathcal{D}_n f(\ell) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + \ (e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)), \\
\mathcal{D}_n f(\ell + \tfrac{n}{2}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell) - \ (e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)), \\
\mathcal{D}_n f(\ell + \tfrac{n}{4}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell + \tfrac{n}{4}) - i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)), \\
\mathcal{D}_n f(\ell + \tfrac{3n}{4}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell + \tfrac{n}{4}) + i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)),
\end{aligned}$$

for $0 \leq \ell \leq \frac{n}{4} - 1$. This gives all the outputs of $\mathcal{D}f(\ell)$ and reduces the number of multiplications and additions required.

Proof by Michael Markiewicz. The first of the four equations comes from what we have shown in part (i):

$$\mathcal{D}_n f(\ell) = \mathcal{D}_{\frac{n}{2}} f_e(\ell) + \left(e^{-2\pi i \ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \right).$$

To show the next three equations, we first observe the following for $m \in \mathbb{Z}$:

$$\begin{aligned}
& e^{-2\pi i(\ell+m\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}((\ell + m\frac{n}{4})) + e^{-2\pi i 3(\ell+m\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}((\ell + m\frac{n}{4})) \\
= \; & e^{-2\pi i(\ell+m\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3(\ell+m\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \\
& \text{(since } \mathcal{D}_{\frac{n}{4}} f_{o,1}, \mathcal{D}_{\frac{n}{4}} f_{o,2} \in l(\mathbb{Z}_{\frac{n}{4}}) \text{, i.e., } n/4 \text{ periodic.)} \\
= \; & e^{-2\pi i \ell/n} e^{-2\pi i m\frac{n}{4}/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n} e^{-2\pi i 3m\frac{n}{4}/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \\
= \; & e^{-2\pi i \ell/n} \left(\cos(-\pi\frac{m}{2}) + i\sin(-\pi\frac{m}{2}) \right) \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) \\
& \qquad + e^{-2\pi i 3\ell/n} \left(\cos(-3\pi\frac{m}{2}) + i\sin(-3\pi\frac{m}{2}) \right) \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \\
= \; & e^{-2\pi i \ell/n} \left(\cos(\pi\frac{m}{2}) - i\sin(\pi\frac{m}{2}) \right) \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) \\
& \qquad + e^{-2\pi i 3\ell/n} \left(\cos(3\pi\frac{m}{2}) - i\sin(3\pi\frac{m}{2}) \right) \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \\
& \text{(since cosine is even and sine is odd).}
\end{aligned}$$

Then for $m = 1$, we have

$$\begin{aligned}
& e^{-2\pi i(\ell+\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}((\ell + \frac{n}{4})) + e^{-2\pi i 3(\ell+\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}((\ell + \frac{n}{4})) \\
= \; & -i \left[e^{-2\pi i \ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i 3\ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \right],
\end{aligned}$$

and for $m = 2$, we have

$$\begin{aligned}
& e^{-2\pi i(\ell+\frac{n}{2})/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}((\ell + \frac{n}{2})) + e^{-2\pi i 3(\ell+\frac{n}{2})/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}((\ell + \frac{n}{2})) \\
= \; & - \left[e^{-2\pi i \ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \right],
\end{aligned}$$

and finally for $m = 3$, we have

$$\begin{aligned}
& e^{-2\pi i(\ell+3\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}((\ell + 3\frac{n}{4})) + e^{-2\pi i 3(\ell+3\frac{n}{4})/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}((\ell + 3\frac{n}{4})) \\
= \; & i \left[e^{-2\pi i \ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i 3\ell/n} \mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell) \right].
\end{aligned}$$

We also know that $\mathcal{D}_{\frac{n}{2}} f_e \in L^2(\mathbb{Z}_{\frac{n}{2}})$ by definition, so

$$\mathcal{D}_{\frac{n}{2}} f_e(\ell + \frac{n}{2}) = \mathcal{D}_{\frac{n}{2}} f_e(\ell),$$

$$\mathcal{D}_{\frac{n}{2}} f_e(\ell + \frac{3n}{4}) = \mathcal{D}_{\frac{n}{2}} f_e(\ell + \frac{n}{4})$$

since it is periodic with period $\frac{n}{2}$.

Using the equalities shown above, it is a direct consequence that the three equations hold:

$$\mathcal{D}_n f(\ell+\frac{n}{2}) = \mathcal{D}_{\frac{n}{2}} f_e(\ell+\frac{n}{2}) + \left(e^{-2\pi i(\ell+\frac{n}{2})/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell+\frac{n}{2}) + e^{-2\pi i 3(\ell+\frac{n}{2})/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell+\frac{n}{2})\right)$$
$$= \mathcal{D}_{\frac{n}{2}} f_e(\ell) - \ (e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)).$$

$$\mathcal{D}_n f(\ell+\frac{n}{4}) = \mathcal{D}_{\frac{n}{2}} f_e(\ell+\frac{n}{4}) + \left(e^{-2\pi i(\ell+\frac{n}{4})/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell+\frac{n}{4}) + e^{-2\pi i 3(\ell+\frac{n}{4})/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell+\frac{n}{4})\right)$$
$$= \mathcal{D}_{\frac{n}{2}} f_e(\ell+\tfrac{n}{4}) - i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)).$$

$$\mathcal{D}_n f(\ell+\frac{3n}{4}) = \mathcal{D}_{\frac{n}{2}} f_e(\ell+\frac{3n}{4}) + \left(e^{-2\pi i(\ell+\frac{3n}{4})/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell+\frac{3n}{4}) + e^{-2\pi i 3(\ell+\frac{3n}{4})/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell+\frac{3n}{4})\right)$$
$$= \mathcal{D}_{\frac{n}{2}} f_e(\ell+\tfrac{n}{4}) + i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)).$$

□

(iii) Explain how the observations in Part (ii) allow you to compute $\mathcal{D}_n f$ from $\mathcal{D}_{\frac{n}{2}} f_e$, $\mathcal{D}_{\frac{n}{4}} f_{o,1}$ and $\mathcal{D}_{\frac{n}{4}} f_{o,2}$ using $6n$ real operations. [Note, multiplications with ± 1 or $\pm i$ do not count, since they amount to negation of real or imaginary parts, which can be absorbed into the next operation by changing it from addition to subtraction or vice versa]

Proof by Michael Markiewicz. Let $0 \le \ell \le \frac{n}{4} - 1$. First we have to compute both $e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell)$ and $e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)$ which takes 12 real operations (since multiplying two complex numbers takes 6 operations and we do that twice).

Next, we can find $e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)$ and $e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i 3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)$ using 2 complex additions for a total of 4 real operations (since a complex addition is equivalent to two real operations).

Finally, we can compute $\mathcal{D}_n f(\ell)$, $\mathcal{D}_n f(\ell+\frac{n}{2})$, $\mathcal{D}_n f(\ell+\frac{n}{4})$, and $\mathcal{D}_n f(\ell+\frac{3n}{4})$ only using one more complex additions each by utilizing the equations we showed in part (ii). Thus, this takes an additional 8 real operations (since a complex addition is equivalent to two real operations).

Therefore, for this particular ℓ, we used 24 real operations. Since we have to do this for $\frac{n}{4}$ different values of ℓ, then in total we used $24 \cdot \frac{n}{4} = 6n$ real operations.

By doing this procedure for every $0 \le l \le \frac{n}{4} - 1$, we compute $\mathcal{D}_n f(\ell)$ from $\mathcal{D}_{\frac{n}{2}} f_e(\ell), \mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell)$, and $\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)$ for all $0 \le \ell \le n-1$ in only $6n$ operations. □

(iv) Show that part (iii) implies that the number of real operations taken by the split-radix FFT, denoted again as A_n, satisfies the recursion

$$A_n = A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + 6n.$$

Explain why $A_1 = 0$ and $A_2 = 4$. Use this to show that $A_n \leq 4n\log_2 n$. [Hint: Define $B_n = A_n - 4n\log_2 n$ and show that B_n satisfies

$$B_n = B_{\frac{n}{2}} + 2B_{\frac{n}{4}}$$

with $B_1 = 0$ and $B_2 = -4$. Use this to argue that $B_n \leq 0$ for all power-of-two n.] [Note: If one is more careful about redundant computations (there are additional multiplications with ± 1 or $\pm i$ that can be skipped), then the complexity of the split-radix FFT algorithm is actually $4n\log_2 n - 6n + 8$ real operations].

Proof by Michael Markiewicz. We define

$$A_n = \text{Number of real operations taken by the split-radix FFT on } L^2(\mathbb{Z}_n).$$

We first note that $A_1 = 0$ since $\mathcal{D}_1$ is the identity. We also note that $A_2 = 4$ since we need to calculate

$$\mathcal{D}_2 f(\ell) = \sum_{k=0}^{1} f(k)\omega^{-k\ell} = f(0) + f(1)\omega^{-\ell}$$

for $\ell = 0, 1$. For $\ell = 0$, we only need 1 complex addition to do $f(0) + f(1)$. For $\ell = 1$, we only need 1 complex addition to do

$$f(0) + f(1)\omega^{-1} = f(0) + f(1)e^{-\pi i} = f(0) - f(1).$$

So in total, we only need 2 complex additions for calculating $\mathcal{D}_2 f$ which equates to 4 real operations.

Thus, at step $m > 2$, we must first compute $\mathcal{D}_{\frac{n}{2}} f_e$, $\mathcal{D}_{\frac{n}{4}} f_{o,1}$, and $\mathcal{D}_{\frac{n}{4}} f_{o,2}$ which take $A_{\frac{n}{2}}$, $A_{\frac{n}{4}}$, and $A_{\frac{n}{4}}$ steps respectively (this is by the definition of A_n). After we calculate those discrete Fourier transforms, we have to compute $\mathcal{D}_n f(\ell), \mathcal{D}_n f(\ell + \frac{n}{2}), \mathcal{D}_n f(\ell + \frac{n}{4})$, and $\mathcal{D}_n f(\ell + \frac{3n}{4})$ for all $0 \leq \ell \leq \frac{n}{4} - 1$ which we have shown takes $6n$ steps.

In total, to calculate the Fourier transform at step m using the Split-Radix FFT, it takes

$$A_{\frac{n}{2}} + A_{\frac{n}{4}} + A_{\frac{n}{4}} + 6n = A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + 6n$$

steps.

Now we define

$$B_n = A_n - 4n\log_2 n.$$

Then

$$B_1 = A_1 - 4(1)\log_2(1) = A_1 = 0$$

and

$$B_2 = A_2 - 4(2)\log_2(2) = A_2 - 8 = -4.$$

We can further show the following about B_n:

$$\begin{aligned}
B_n &= A_n - 4n\log_2 n \\
&= A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + 6n - 4n\log_2 n \\
&= A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + (2n + 4n) - 4n\log_2 n \\
&= A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + (2n\log_2(2) + 2n\log_2(4)) - 4n\log_2 n \\
&= A_{\frac{n}{2}} + 2A_{\frac{n}{4}} - (2n\log_2 n - 2n\log_2(2)) - (2n\log_2 n - 2n\log_2(4)) \\
&= A_{\frac{n}{2}} + 2A_{\frac{n}{4}} - 2n\log_2(\frac{n}{2}) - 2n\log_2(\frac{n}{4}) \\
&= \left(A_{\frac{n}{2}} - 2n\log_2(\frac{n}{2})\right) + 2\left(A_{\frac{n}{4}} - n\log_2(\frac{n}{4})\right) \\
&= \left(A_{\frac{n}{2}} - 4(\frac{n}{2})\log_2(\frac{n}{2})\right) + 2\left(A_{\frac{n}{4}} - 4(\frac{n}{4})\log_2(\frac{n}{4})\right) \\
&= B_{\frac{n}{2}} + 2B_{\frac{n}{4}}.
\end{aligned}$$

We show by strong mathematical induction that $B_n \leq 0$ for all powers of 2. We have already shown the base cases of $B_1 = 0$ and $B_2 = -4$ so we move to the inductive step. Assume $B_m \leq 0$ for all powers of 2 less k where k is some power of 2. Then for B_k, we have

$$B_k = B_{\frac{k}{2}} + 2B_{\frac{k}{4}}.$$

Since we assumed that all powers of 2 less than k were nonpositive, then $B_{\frac{k}{2}} \leq 0$ and $B_{\frac{k}{2}} \leq 0$ and the sum of nonpositive numbers is also nonpositive. Therefore, $B_k \leq 0$.

This completes our proof by mathematical induction that $B_n \leq 0$ for all powers of 2. This also implies that $A_n - 4n\log_2 n \leq 0 \implies A_n \leq 4n\log_2 n$ by definition of B_n. Thus, the complexity of the split-radix FFT is at most $4n\log_2 n$ real operations. □

2. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

   (i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

   $$\nabla^- f(k) = f(k) - f(k-1).$$

   Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = g * f$ and use the DFT convolution property $\mathcal{D}(g * f) = \mathcal{D}g\mathcal{D}f$ to show that $\mathcal{D}(\nabla^- f)(k) = (1 - \omega^{-k})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

   Proof by Dingjun Bian. We define $g \in L^2(\mathbf{Z}_n)$ to be

   $$g(x) = \begin{cases} 1, & \text{when x = 0} \\ -1, & \text{when x = 1} \\ 0, & \text{otherwise.} \end{cases}$$

Then we must have

$$\begin{aligned} g * f(k) &= \sum_{j=0}^{n-1} g(j) f(k-j) \\ &= g(0) f(k) + g(1) f(k-1) \\ &= f(k) - f(k-1) \\ &= \nabla^{-} f(k) \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{D}(\nabla^{-} f)(k) &= \mathcal{D}(g * f(k)) \\ &= \mathcal{D}g(k)\mathcal{D}f(k) \\ &= \left(\sum_{l=0}^{n-1} g(l) e^{\frac{-2\pi i k l}{n}} \right) \mathcal{D}f(k) \\ &= (1 - e^{-\frac{2\pi i k}{n}})\mathcal{D}f(k) \\ &= (1 - \omega^{-k})\mathcal{D}f(k), \end{aligned}$$

where $\omega = e^{\frac{2\pi i}{n}}$. Therefore, we have proven the desired result. □

(ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

$$\nabla^{+} f(k) = f(k+1) - f(k).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^{+} f = g * f$ use this to show that $\mathcal{D}(\nabla^{+} f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

Proof by Dingjun Bian. We define $g \in L^2(\mathbf{Z}_n)$ such that

$$g(x) = \begin{cases} -1, & \text{when x = 0} \\ 1, & \text{when x = n - 1} \\ 0, & \text{otherwise.} \end{cases}$$

Then we must have

$$\begin{aligned} g * f(k) &= \sum_{j=0}^{n-1} g(j) f(k-j) \\ &= g(0) f(k) + g(n-1) f(k-n+1) \\ &= -f(k) + f(k+1-n) \\ &= f(k+1) - f(k) \\ &= \nabla^{+} f(k) \end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathcal{D}(\nabla^+ f)(k) &= \mathcal{D}(g * f(k)) \\ &= \mathcal{D}g(k)\mathcal{D}f(k) \\ &= \left(\sum_{l=0}^{n-1} g(l) e^{\frac{-2\pi i k l}{n}}\right)\mathcal{D}f(k) \\ &= (-1 + e^{-\frac{2\pi i k(n-1)}{n}})\mathcal{D}f(k) \\ &= (e^{\frac{2\pi i k}{n}} - 1)\mathcal{D}f(k) \\ &= (\omega^k - 1)\mathcal{D}f(k),\end{aligned}$$

where $\omega = e^{\frac{2\pi i}{n}}$. Therefore, we have proven the desired result. □

(iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

$$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i\sin(2\pi k/n)\mathcal{D}f(k).$$

Proof by Dingjun Bian. We note that

$$\begin{aligned}\mathcal{D}(\nabla f)(k) &= \mathcal{D}(\frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k))) \\ &= \sum_{l=0}^{n-1} \frac{1}{2}(\nabla^- f(l) + \nabla^+ f(l))\omega^{kl} \\ &= \frac{1}{2}\left(\sum_{l=0}^{n-1} \nabla^- f(l)\omega^{kl} + \sum_{l=0}^{n-1} \nabla^+ f(l)\omega^{kl}\right) \\ &= \frac{1}{2}\left(\mathcal{D}(\nabla^+ f)(k) + \mathcal{D}(\nabla^- f)(k))\right) \\ &= \frac{1}{2}\left((\omega^k - 1)\mathcal{D}f(k) + (1 - \omega^{-k})\mathcal{D}f(k)\right) \\ &= \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) \\ &= \frac{1}{2}(e^{\frac{2\pi i k}{n}} - e^{-\frac{2\pi i k}{n}})\mathcal{D}f(k) \\ &= \frac{1}{2}\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} - \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)\mathcal{D}f(k) \\ &= i\sin\frac{2\pi k}{n}\mathcal{D}f(k).\end{aligned}$$

Therefore, we have proven the desired result. □

(iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

$$\Delta f(k) = \nabla^+\nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k).$$

Proof by Dingjun Bian. We note that

$$\begin{aligned}
\mathcal{D}(\Delta f)(k) &= \mathcal{D}(\nabla^+\nabla^- f)(k) \\
&= (\omega^k - 1)\mathcal{D}(\nabla^- f)(k) \\
&= (\omega^k - 1)(1 - \omega^{-k})\mathcal{D}f(k) \\
&= (\omega^k - \omega^{k-k} - 1 + \omega^{-k})\mathcal{D}f(k) \\
&= (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) \\
&= \left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} + \cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n} - 2\right)\mathcal{D}f(k) \\
&= 2\left(\cos\frac{2\pi k}{n} - 1\right)\mathcal{D}f(k).
\end{aligned}$$

Therefore, we have proven the desired result. □

3. Consider the Poisson equation

$$\Delta u = f \quad \text{on } \mathbb{Z}_n. \tag{2}$$

The source term $f \in L^2(\mathbb{Z}_n)$ is given, and $u \in L^2(\mathbb{Z}_n)$ is the unknown we wish to solve for. The discrete Laplacian Δ is defined in Problem 2. Use the DFT and the results from Problem 2 to derive a solution formula for u using one forward transform $\mathcal{D}$ and one inverse transform $\mathcal{D}^{-1}$. Is there a condition you need to place on $\mathcal{D}f$ for your solution formula to make sense? [Hint: Take the DFT of both sides of (2), solve for $\mathcal{D}u$, and then apply the inverse DFT $\mathcal{D}^{-1}$. Be careful not to divide by zero when you solve for $\mathcal{D}u$.]

Proof. Using the results in Part 2(iv) we take the DFT on both sides of the equation to obtain

$$2(\cos(2\pi k/n) - 1)\mathcal{D}u(k) = \mathcal{D}f(k). \tag{3}$$

When $k = 0$, the left hand size vanishes, so $\mathcal{D}f(0) = 0$ is a necessary condition for the existence of a solution. This means that

$$0 = \mathcal{D}f(0) = \sum_{j=0}^{n-1} f(j).$$

Thus, the function f must have mean value zero. Assuming this is the case, we can solve for $\mathcal{D}u(k)$ in (3) for $k \geq 1$, yielding

$$\mathcal{D}u(k) = \frac{\mathcal{D}f(k)}{2(\cos(2\pi k/n) - 1)}.$$

To write an expression that holds for all $k \geq 0$, we define

$$G(k) = \begin{cases} \frac{1}{2(\cos(2\pi k/n)-1)}, & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

Then we have $\mathcal{D}u(k) = G(k)\mathcal{D}f(k)$ for all k, and hence by the convolution theorem we have

$$u = g * f$$

solves the Poisson equation (2), where $g = \mathcal{D}^{-1}G$. This solution satisfies $\mathcal{D}u(0) = 0$, but noting (3), the value of $\mathcal{D}u(0)$ does not enter into the equation, so we may set it arbitrarily. Since

$$\mathcal{D}u(0) = \sum_{j=0}^{n-1} u(j),$$

this amounts to setting the mean value of u arbitrarily. Thus, the most general form for the solution of (2) is

$$u = C + g * f,$$

where $C \in \mathbb{R}$ is an arbitrary constant. In this case $\mathcal{D}u(0) = Cn$. □

4. Let $n \geq 1$ be odd. Show that for $t \notin \mathbb{Z}$ we have

$$\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi ikt} = \frac{\operatorname{sinc}(nt)}{\operatorname{sinc}(t)}.$$

What happens when $t \in \mathbb{Z}$? Here, sinc is the normalized sinc function $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

Proof by Eduardo Torres Davilla. Let's begin by showing for any $t \notin \mathbb{Z}$ we have

$$\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi ikt} = \frac{\operatorname{sinc}(nt)}{\operatorname{sinc}(t)}$$

where $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. First let's try to rewrite the summation on the left hand side so that it's easier to work with. Let's define $m = \frac{n-1}{2}$ and $r = e^{2\pi it}$ which gives us

$$\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi ikt} = \frac{1}{n} \sum_{k=-m}^{m} r^k.$$

Now let $S_m = \sum_{k=-m}^{m} r^k$ and we notice that the following holds

$$\begin{aligned} r \cdot S_m - S_m &= r \cdot \sum_{k=-m}^{m} r^k - \sum_{k=-m}^{m} r^k \\ &= \sum_{k=-m}^{m} r^{k+1} - \sum_{k=-m}^{m} r^k \\ &= r^{-m+1} + r^{-m+2} + \cdots + r^m + r^{m+1} - r^{-m} - r^{-m+1} - \cdots - r^m \\ &= r^{m+1} - r^{-m} \end{aligned}$$

thus showing us that

$$\begin{aligned}
r \cdot S_m - S_m &= r^{m+1} - r^{-m} \\
\iff \quad S_m(r-1) &= r^{m+1} - r^{-m} \\
\iff \quad S_m &= \frac{r^{m+1} - r^{-m}}{r-1} \\
\iff \quad S_m &= \left(\frac{r^{1/2}}{r^{1/2}}\right) \frac{r^{m+(1/2)} - r^{-m-(1/2)}}{r^{1/2} - r^{-1/2}}.
\end{aligned}$$

Now let's continue by substituting back $r = e^{2\pi i t}, m = \frac{n-1}{2}$, and use the identity of $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$ which gives us the following

$$\begin{aligned}
\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i k t} &= \frac{1}{n} \sum_{k=-m}^{m} r^k \\
&= \frac{1}{n} \left(\frac{r^{1/2}}{r^{1/2}}\right) \frac{r^{m+(1/2)} - r^{-m-(1/2)}}{r^{1/2} - r^{-1/2}} \\
&= \frac{1}{n} \left(\frac{e^{2\pi i t(m+(1/2))} - e^{-2\pi i t(m+(1/2))}}{e^{\pi i t} - e^{-\pi i t}}\right) \\
&= \frac{1}{n} \left(\frac{2i\sin(2\pi t(m+(1/2)))}{2i\sin(\pi t)}\right) \\
&= \frac{1}{n} \left(\frac{\sin(2\pi t((n-1)/2 + (1/2)))}{\sin(\pi t)}\right) \\
&= \frac{\sin(n\pi t)}{n\sin(\pi t)} \\
&= \frac{\sin(n\pi t)}{n\pi t} \cdot \frac{\pi t}{\sin(\pi t)} \\
&= \frac{\frac{\sin(n\pi t)}{n\pi t}}{\frac{\sin(\pi t)}{\pi t}} \\
&= \frac{\operatorname{sinc}(n\pi t)}{\operatorname{sinc}(\pi t)}
\end{aligned}$$

giving us the desired equality.

Now, we continue to show what happens when $t \in \mathbb{Z}$. If $t \in \mathbb{Z}$ we have the following on the left hand side of the equality

$$\begin{aligned}
\frac{1}{n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi ikt} &= \frac{1}{n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \cos(2\pi kt) + i\sin(2\pi kt) \\
&= \frac{1}{n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} 1 + i\cdot 0 \\
&= \frac{1}{n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} 1 \\
&= 1
\end{aligned}$$

since $\cos(2\pi\ell) = 1$ for any $\ell \in \mathbb{Z}$ and $\sin(2\pi\ell) = 0$ for any $\ell \in \mathbb{Z}$. Now, moving on to the right hand side, we have the following

$$\begin{aligned}
\frac{\operatorname{sinc}(n\pi t)}{\operatorname{sinc}(\pi t)} &= \frac{\frac{\sin(n\pi t)}{n\pi t}}{\frac{\sin(\pi t)}{\pi t}} \\
&= \frac{\sin(n\pi t)}{n\pi t}\cdot\frac{\pi t}{\sin(\pi t)} \\
&= \frac{\sin(n\pi t)}{n\sin(\pi t)} \\
&= \frac{0}{0}
\end{aligned}$$

which is undefined thus the equality does not work if $t \in \mathbb{Z}$. $\square$