Mathematics of Image and Data Analysis
Math 5467

The Discrete Fourier Transform

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Last time

- Newton's Method

Today

- Discrete Fourier Transform (DFT)

Audio compression basis

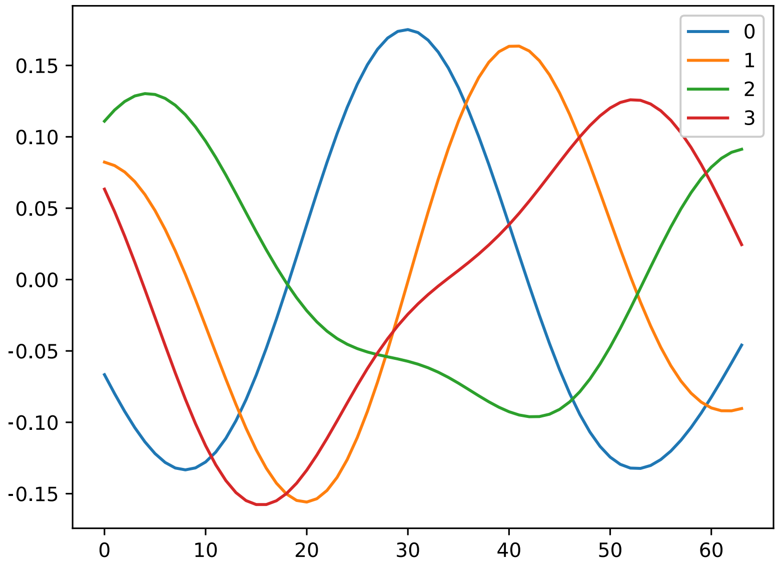


Figure 1: The first 4 principal components computed during PCA-based audio compression. Two of the basis functions strongly resemble the trigonometric functions sin and cos.

A role for a hand-crafted change of basis

- PCA finds the best change of basis that represents your data with as few basis vectors as possible.
- In some setting PCA is too expensive (embedded environments, cell phones, digital cameras, video surveillance, etc.).
- A hand-crafted change of basis can be computed very efficiently and studied much more deeply mathematically.

Complex numbers

We recall that a complex number has the form $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of all complex numbers is denoted $\mathbb{C}$. For a complex number $z = a + ib$, the *complex conjugate*, denoted $\overline{z}$, is given by

$$\overline{z} = a - ib.$$

The *modulus* of z, denoted $|z|$, is given by

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}.$$

Complex exponential and Euler's formula

The complex exponential of $z \in \mathbb{C}$ is defined by the Taylor series expansion

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

The Taylor series is absolutely convergent in the whole complex plane. A very important identity involving the complex exponential is Euler's identity

$$e^{it} = \cos t + i \sin t \tag{1}$$

for all real numbers $t \in \mathbb{R}$.

Proof of Euler's formula

$$f(t) = \cos(t) + i\sin(t)$$

$$f'(t) = -\sin(t) + i\cos(t)$$

$$= i\left(\cos(t) - \frac{1}{i}\sin(t)\right)$$

$$\frac{1}{i} = \frac{1}{i}\left(\frac{i}{i}\right) = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$= i\left(\cos(t) + i\sin(t)\right)$$

$$= i f(t)$$

$$\frac{d}{dt}\left(\frac{f(t)}{e^{it}}\right) = \frac{e^{it} f'(t) - f(t)\, i e^{it}}{(e^{it})^2}$$

$$= \frac{\cancelto{0}{e^{it} i f(t) - f(t)\, i e^{it}}}{e^{2it}}$$

$$= 0$$

$$\Rightarrow \quad f(t) = C e^{it}$$

$$f(0) = C, \quad f(0) = 1$$

$$\Rightarrow C = 1 \qquad \blacksquare$$

The Discrete Fourier Transform (DFT)

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$ (i.e., integers $p, q \in \mathbb{Z}_n$ are added, subtracted, or multiplied, the result is interpreted modulo n).

Example 1. In $\mathbb{Z}_4$ we have $2 + 2 = 4 = 0 \bmod 4$. △

$\mathbb{C}^n \approx \mathbb{R}^{2n}$

Let $L^2(\mathbb{Z}_n)$ denote the vector space of functions $f : \mathbb{Z}_n \to \mathbb{C}$. We define the inner product on $L^2(\mathbb{Z}_n)$ by

$$\langle f, g \rangle = \sum_{k=0}^{n-1} f(k)\overline{g(k)}.$$

The norm of $f \in L^2(\mathbb{Z}_n)$ is defined by $\|f\| = \sqrt{\langle f, f \rangle}$.

$f(k)\overline{f(k)} = |f(k)|^2$

The Discrete Fourier Transform (DFT)

The DFT is an orthogonal change of basis in $L^2(\mathbb{Z}_n)$ that expresses a function $f \cancel{:} \in L^2(\mathbb{Z}_n) \to \mathbb{C}$ in terms sinusoidal basis functions of different frequencies

$$k \mapsto e^{2\pi i \sigma k} = \cos(2\pi\sigma k) + i \sin(2\pi\sigma k). \tag{2}$$

Which frequencies?

$$\text{Period} = T = \frac{1}{\sigma}$$

$$\text{Freq} = \sigma$$

Periods should divide evenly into n to ensure n-periodic

$$n = \ell T, \quad \ell \in \mathbb{Z}_+$$

$$n = \frac{\ell}{\sigma} \Rightarrow \sigma = \frac{\ell}{n}$$

$$e^{2\pi i \frac{\ell}{n} k} \overset{\ell = n}{=} e^{2\pi i k}, \quad k = 0, \dots, n-1$$

$$= \underbrace{\cos(2\pi k)}_{=1} + i \underbrace{\sin(2\pi k)}_{=0}$$

$$= 1$$

$$= e^{2\pi i \frac{0}{n} k}$$

$$\ell = 0, 1, \dots, n-1$$

DFT basis functions

$\sigma = \frac{\ell}{n}$

$k = 0, 1, \dots, n-1$

We define

(3) $$u_\ell(k) := e^{2\pi i k\ell/n}, \quad \ell = 0, 1, \dots, n-1.$$

It is often useful to note that we can set $\omega = e^{2\pi i/n}$ and write

$$u_\ell(k) = \omega^{k\ell}.$$

The complex number ω is an n^{th} root of unity, meaning that

$$\omega^n = e^{2\pi i} = 1.$$

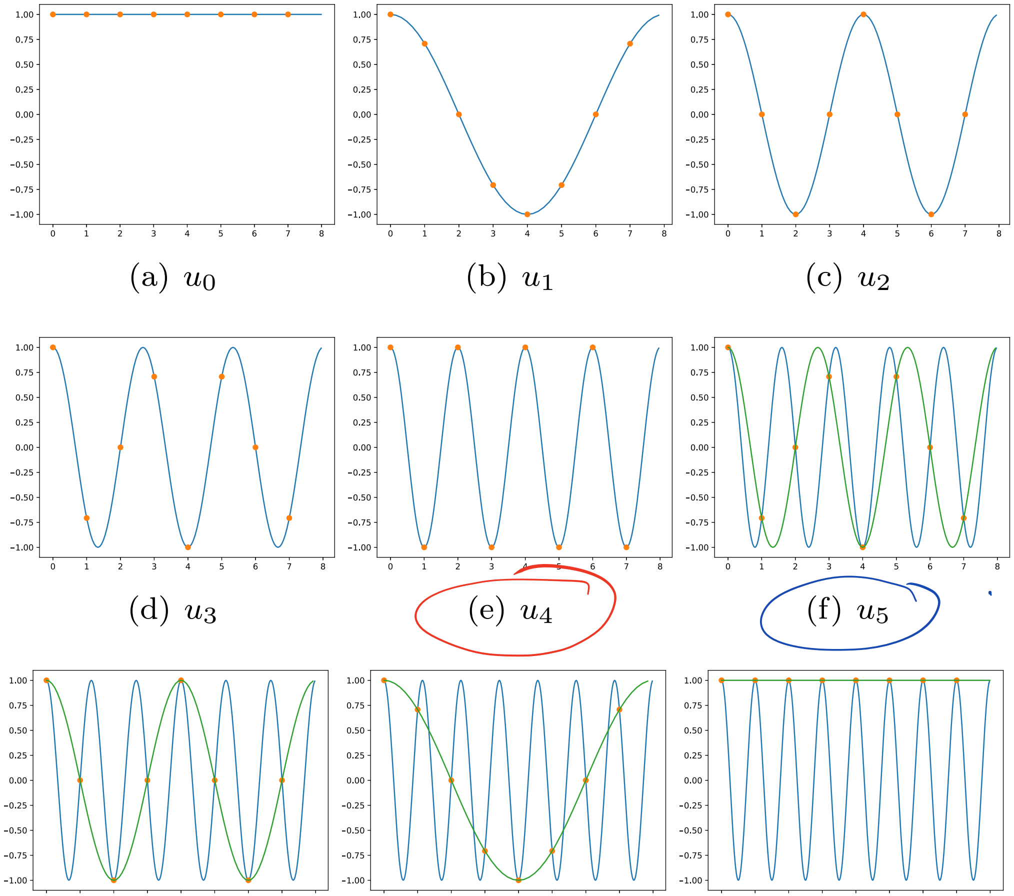
We also have $\overline{\omega} = e^{-2\pi i/n} = \omega^{-1}$.

$$\overline{\omega} = \overline{\left(\cos(2\pi/n) + i\sin\left(\frac{2\pi}{n}\right)\right)}$$

$$= \cos\left(\frac{2\pi}{n}\right) - i\sin\left(\frac{2\pi}{n}\right)$$

$$= \cos\left(-\frac{2\pi}{n}\right) + i\sin\left(-\frac{2\pi}{n}\right) = e^{-\frac{2\pi i}{n}}$$

$= \omega^{-1}$



(a) u_0 (b) u_1 (c) u_2

(d) u_3 (e) u_4 (f) u_5

(g) u_6 $\ell = 2$ (h) u_7 $n = 8$ $\ell = 1$ (i) u_8

Aliasing

$$u_{n-\ell}(k) = \omega^{k(n-\ell)}$$

$$= \omega^{kn}\,\omega^{-\ell k}$$

$$= \underbrace{(\omega^{n})^{k}}_{=1}\,\omega^{-\ell k}$$

$$\overline{\omega} = \omega^{-1}$$

$$= \overline{(\omega^{\ell k})}$$

$$= \overline{u_{\ell}(k)}$$

$u_{n-\ell}$ aliases to $\overline{u_{\ell}(k)}$

Highest frequency is $\ell = \frac{n}{2}$.

Orthogonality

Lemma 1. *The functions* $u_0, u_1, \ldots, u_{n-1}$ *are orthogonal. In particular*

$$\langle u_\ell, u_m \rangle = \begin{cases} n, & \text{if } \ell = m \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Proof:

$$\langle u_\ell, u_m \rangle = \sum_{k=0}^{n-1} u_\ell(k)\, \overline{u_m(k)}$$

$$= \sum_{k=0}^{n-1} w^{\ell k} w^{-mk}$$

$$= \sum_{k=0}^{n-1} w^{(\ell-m)k}$$

$$\boxed{\begin{aligned} u_\ell(k) &= w^{\ell k} \\ w &= e^{2\pi i/n} \\ \overline{w} &= w^{-1} \end{aligned}}$$

So $\langle u_\ell, u_\ell \rangle = n$. Assume $\ell \neq m$

Set $r = w^{\ell - m}$ so that $r^n = 1$

$$\langle u_\ell, u_m \rangle = \sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r} \overset{(*)}{=} 0$$

Geometric Series

$$S_n = \sum_{k=0}^{n-1} r^k$$

$$r S_n = \sum_{k=0}^{n-1} r^{k+1} = r^n + \sum_{k=1}^{n-1} r^k$$

$$= r^n + S_n - 1$$

$$r S_n - S_n = r^n - 1 \Rightarrow S_n = \frac{r^n - 1}{r - 1}$$

(*) because $r = \omega^{l-m}$

$$r^n = (\omega^n)^{l-m} = 1^{l-m} = 1$$

since $\omega^n = (e^{2\pi i/n})^n = 1$

$\blacksquare$

How do we express $f \in L^2(\mathbb{Z}_n)$

$$f(k) = \sum_{l=0}^{n-1} c_l u_l(k)$$

$$\langle f, u_m \rangle = \left\langle \sum_{l=0}^{n-1} c_l u_l, u_m \right\rangle$$

$$= \sum_{l=0}^{n-1} c_l \underbrace{\langle u_l, u_m \rangle}$$

$$= \begin{cases} n, & l = m \\ 0, & l \neq m \end{cases}$$

$$= n c_m$$

$$c_m = \frac{1}{n} \langle f, u_m \rangle$$

$$f = \frac{1}{n} \sum_{l=0}^{n-1} \langle f, u_l \rangle u_l$$

Definition

Definition 2. The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \underbrace{\sum_{k=0}^{n-1} f(k)\omega^{-k\ell}}_{=\langle f, u_\ell \rangle} = \sum_{k=0}^{n-1} f(k)e^{-2\pi i k\ell/n},$$

where $\omega = e^{2\pi i/n}$.

$u_\ell(k) = \omega^{k\ell}$

Proposition 3. *If $f \in L^2(\mathbb{Z}_n)$ is real-valued (i.e., $f(k) \in \mathbb{R}$ for all k), then*

$$\mathcal{D}f(\ell) = \overline{\mathcal{D}f(n-\ell)}.$$

aliasing

Proof: $\mathcal{D}f(\ell) = \langle f, u_\ell \rangle = \langle f, \overline{u_{n-\ell}} \rangle$

$$= \sum_{k=0}^{n-1} f(k)\, u_{n-\ell}(k)$$

$$= \overline{\sum_{k=0}^{n-1} f(k)\, \overline{u_{n-\ell}(k)}}$$

$$= \overline{\langle f, u_{n-\ell} \rangle}$$

$$= \overline{Df(n-\ell)}$$

Inverse Fourier Transform

Theorem 4 (Fourier Inversion Theorem)**.** *For any $f \in L^2(\mathbb{Z}_n)$ we have*

$$f(k) = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)e^{2\pi i k\ell/n}. \tag{5}$$

Definition 5 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{2\pi i k\ell/n}.$$

$= \omega^{k\ell}$

$$f = \frac{1}{n} \sum_{\ell=0}^{n-1} \underbrace{\langle f, u_\ell \rangle}_{Df(\ell)} \widetilde{u_\ell}$$

Matrix version

Remark 6. Define the $n \times n$ complex-valued matrix with entries $W(k,\ell) = \omega^{k\ell}$, that is

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \tag{6}$$

Then the DFT can be expressed via matrix multiplication as $\mathcal{D}f = \overline{W}f$. The inverse DFT can be expressed as $\mathcal{D}^{-1}f = \frac{1}{n}Wf$. In both cases we treat f as a vector $f \in \mathbb{C}^n$. Theorem 4 (Fourier Inversion) can be restated as saying that $W\overline{W} = nI$.

Basic properties

Exercise 7. Show that the DFT enjoys the following basic shift properties.

1. Recall that $u_\ell(k) := e^{2\pi i k\ell/n} = \omega^{k\ell}$. Show that

$$\mathcal{D}(f \cdot u_\ell)(k) = \mathcal{D}f(k+\ell).$$

2. Let $T_\ell : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ be the translation operator $T_\ell f(k) = f(k-\ell)$. Show that

$$\mathcal{D}(T_\ell f)(k) = e^{-2\pi i k\ell/n}\mathcal{D}f(k).$$

   [Hint: You can equivalently show that $\mathcal{D}^{-1}(f \cdot u_\ell)(k) = \mathcal{D}^{-1}f(k-\ell)$, using an argument similar to part 1.]

$\triangle$

Intro to DFT (.ipynb)