Mathematics of Image and Data Analysis
Math 5467

Linear Algebra, Calculus & Python I

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Announcements

- HW1 due Friday. Submission will be via Google Forms. Details to come on Wednesday.
  - Code as Google Colab Notebook (or .py or .ipynb files). Half the marks for programming exercises or projects are awarded if your code runs without errors. Make sure your code works!
    - Do a factory reset in Google Colab and test your code on a clean kernel.
  - Math can be included in a Google Colab Notebook (it supports LaTeX), or typed up in LaTeX (submit a PDF), or handwritten and scanned (high quality) with scanner or smartphone app.
- 3 choices for project 1 are up on the website. Choose 1 to complete.

  http://www-users.math.umn.edu/~jwcalder/5467/homework.html
  - Project descriptions are in the course notes, and Python notebooks on course website.
- I will answer questions nightly on Piazza.

Last time: Linear algebra review

- Capital letters A, B, C for matrices (entries are $A(i, j)$)
- Lower case letteers $x, y, z, x_1, x_2, x_3, x_4, \ldots$ for (column) vectors.
- $e_1, e_2, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
- Matrix multiplication: A is $m \times n$ and B is $n \times p$ then $C = AB$ is the $m \times p$ matrix with entries
$$C(i, j) = \sum_{k=1}^{n} A(i, k)B(k, j).$$
- A^T denotes the transpose of A.
- Dot product $x^T y = \sum_{i=1}^{n} x(i)y(i)$.
- Norm: $\|x\| = \sqrt{x^T x} = \sqrt{x(1)^2 + x(2)^2 + \cdots + x(n)^2}$.
- Algebra: $\|x \pm y\|^2 = \|x\|^2 \pm 2x^T y + \|y\|^2$.

Rank-one matrix

For vectors x, y of length n, the rank-one matrix $A = xy^T$ is the $n \times n$ matrix with entries

$$A(i,j) = x(i)y(j).$$

It is called rank-one since the range of A is one dimensional and spanned by the vector x. Indeed,

$$Az = xy^T z = (y^T z)x$$

for any vector z.

Exercise

Let $x_1, x_2, x_3, \ldots, x_m$ be a collection of vectors of length n. Define the $m \times n$ matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{bmatrix}.$$

Show that

$$\sum_{i=1}^{m} x_i x_i^T = X^T X.$$

Today

- Projection
- Introduction to Numpy

Projection

Let $L \subset \mathbb{R}^n$ be a linear subspace spanned by the orthonormal vectors $v_1, v_2, \ldots, v_p$, where $p \le n$. That is

$$L = \left\{ \sum_{i=1}^{p} a_i v_i \,:\, a_i \in \mathbb{R} \right\}.$$

In this case, L is p-dimensional.

- Orthonormal means that $\|v_i\| = 1$ and $v_i^T v_j = 0$ for $i \neq j$.

$1 = \|v_i\| = \sqrt{v_i^T v_i}$, $v_i^T v_i = 1$

Definition 1. The *projection* of a point $x \in \mathbb{R}^n$ onto L, denoted $\mathrm{Proj}_L x$, is the closest point in the subspace L to x. That is, $\mathrm{Proj}_L x \in L$ satisfies

$$\|\mathrm{Proj}_L x - x\| \le \|y - x\| \quad \text{for all } y \in L.$$

Projection

We claim that

$$\text{Proj}_L x = \sum_{i=1}^{p} \overbrace{(x^T v_i)}^{a_i} v_i$$

Write $z = \text{proj}_L x = \sum_{i=1}^{p} a_i v_i$

$$\|\text{proj}_L x - x\|^2 = \|z - x\|^2$$

$$= \|z\|^2 - 2x^T z + \|x\|^2$$

$$\|z\|^2 = z^T z = z^T \sum_{i=1}^{p} a_i v_i$$

$$= \sum_{i=1}^{p} a_i z^T v_i$$

$$= \sum_{i=1}^{p} a_i \left(\sum_{j=1}^{p} a_j v_j\right)^T v_i$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} a_i a_j v_j^T v_i$$

orthonormal

$v_j^T v_i = 0$ if $i \neq j$

$v_i^T v_i = 1$

$$\|z\|^2 = \sum_{i=1}^{p} a_i^2$$

$$x^T z = x^T \sum_{i=1}^{p} a_i v_i = \sum_{i=1}^{p} a_i x^T v_i$$

$$\|\mathrm{proj}_L x - x\|^2 = \sum_{i=1}^{p} a_i^2 - 2\sum_{i=1}^{p} a_i x^T v_i + \|x\|^2$$

$$= \sum_{i=1}^{p} \left(a_i^2 - 2 a_i x^T v_i\right) + \|x\|^2$$

Minimize over a_i. Diff. in a_i

set $\frac{\partial}{\partial a_i} \| \mathrm{proj}_L x - x \|^2 = 0$

$$\cancel{2} a_i - \cancel{2} x^T v_i = 0$$

$$\boxed{a_i = x^T v_i}$$

Projection

Since the v_i are orthonormal, we have

$$\|\mathrm{Proj}_L x\|^2 = \sum_{j=1}^{p} (x^T v_i)^2 \tag{1}$$

Recall $z = \sum_{i=1}^{p} a_i v_i$, $\|z\|^2 = \sum_{i=1}^{p} a_i^2$

$$\mathrm{Proj}_L x = \sum_{i=1}^{p} (v_i^T x) v_i$$

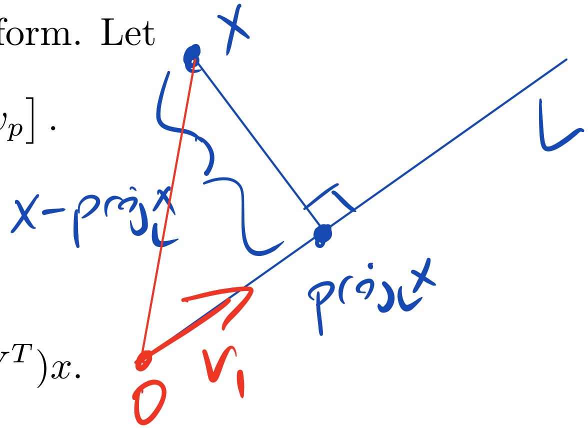
$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}}_{} \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_p^T x \end{bmatrix}$$

V

$$= V \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_p^T \end{bmatrix} x = V V^T x$$

Projection

It can be useful to write the projection in matrix form. Let



$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}.$$

Then we have

$$\mathrm{Proj}_L x = VV^T x.$$

The *residual* is

$$x - \mathrm{Proj}_L x = (I - VV^T)x.$$

Exercise 2. Show that the projection is orthogonal, that is

$$(x - \mathrm{Proj}_L x)^T v_i = 0, \quad i = 1, \dots, p.$$

Use this to show that

$$\|x\|^2 = \|\mathrm{Proj}_L x\|^2 + \|x - \mathrm{Proj}_L x\|^2.$$

Pythagorean Theorem

△

$$(x - \mathrm{proj}_L x)^T v_j = \left(x - \sum_{i=1}^{p} (v_i^T x) v_i\right)^T v_j$$

$$= \left(x^T - \sum_{i=1}^{p} (v_i^T x) v_i^T\right) v_j$$

$$= x^T v_j - \sum_{i=1}^{p} (v_i^T x) \underbrace{v_i^T v_j}$$

$= 0$ if $i \neq j$

$= 1$ if $i = j$

$$= x^T v_j - v_j^T x$$

$$= 0$$

for every v_j, $j = 1, \dots, p$.

$$\|x\|^2 = \|\mathrm{proj}_L x\|^2 + \|x - \mathrm{proj}_L x\|^2$$

$z = \mathrm{proj}_L x$

$$\|\mathrm{proj}_L x - x\|^2 = \|z - x\|^2$$

$$= \|z\|^2 - 2x^T z + \|x\|^2$$

$$= \|\mathrm{proj}_L x\|^2 - 2x^T \mathrm{proj}_L x + \|x\|^2$$

$$= \sum_{i=1}^{p} (x^T v_i)^2 - 2x^T \sum_{i=1}^{p} (x^T v_i) v_i + \|x\|^2$$

$$= \sum_{i=1}^{p} (x^T v_i)^2 - 2\sum_{i=1}^{p} \underbrace{(x^T v_i)(x^T v_i)}_{(x^T v_i)^2} + \|x\|^2$$

$$= -\sum_{i=1}^{p} (x^T v_i)^2 + \|x\|^2$$

$$= -\|\mathrm{proj}_L x\|^2 + \|x\|^2$$

$\blacksquare$

$$V = [v_1 \cdots v_p]$$

Exercises

Exercise 3. Show that

(i) $V^TV = I$.

(ii) $(VV^T)^2 = VV^T$

(iii) $(I - VV^T)^2 = I - VV^T$.

$$(VV^T)^2 = V\underbrace{(V^TV)}_{I}V^T = VV^T$$

$$V^TV = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_p^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$$

$$= \begin{bmatrix} v_1^Tv_1 & v_1^Tv_2 & \cdots & v_1^Tv_p \\ v_2^Tv_1 & v_2^Tv_2 & & \vdots \\ \vdots & & \ddots & \\ v_p^Tv_1 & \cdots & & v_p^Tv_p \end{bmatrix}$$

$$= I$$

△

Exercise 4. Let L be a linear subspace of $\mathbb{R}^n$.

(i) Show that $\|\mathrm{Proj}_L x\| \le \|x\|$.

(ii) Show that $\mathrm{Proj}_L x = x$ if and only if $x \in L$.

(iii) Show that if $\mathrm{Proj}_L x = x$ for all $x \in \mathbb{R}^n$, then $L = \mathbb{R}^n$.

△

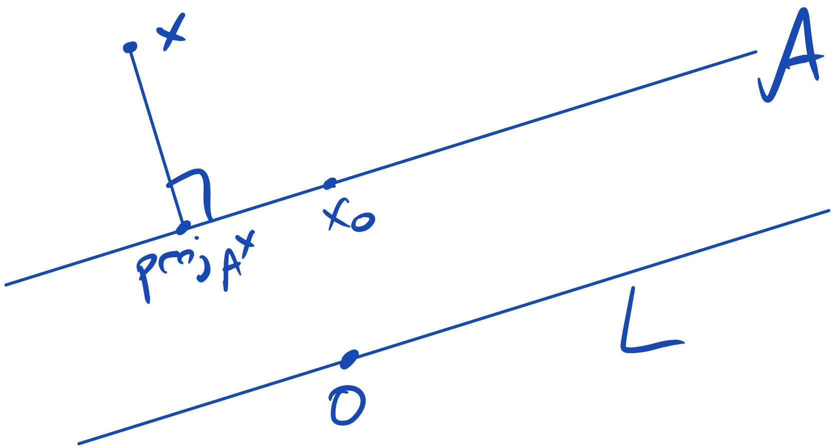
Affine projection

An affine space has the form

$$A = x_0 + L = \{x_0 + y \ : \ y \in L\}$$

where L is a linear subspace of $\mathbb{R}^n$. Projection onto an affine space is given by

$$\mathrm{Proj}_A x = x_0 + \mathrm{Proj}_L(x - x_0). \tag{2}$$



Introduction to Numpy(.ipynb)