Mathematics of Image and Data Analysis
Math 5467

The Sampling Theorem and Cosine Transform

Instructor: Jeff Calder
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/5467

Last time

- Mult-dimensional DFT
- Image denoising

Today

- The sampling theorem
- Discrete Cosine Transform

The Sampling Theorem

If a signal $u : \mathbb{R} \to \mathbb{R}$ contains no frequencies greater than σ_{max}, then u can be perfectly reconstructed from evenly spaced samples provided the sampling frequency is greater than the Nyquist rate $2\sigma_{max}$ and we have the Sinc Interpolation formula

$$u(t) = \sum_{j=-\infty}^{\infty} u(jh) \operatorname{sinc}\left(\frac{t - jh}{h}\right), \tag{1}$$

where h is the sampling period and

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

The Sampling Theorem

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$$u(t) = \sum_{j=-\infty}^{\infty} u(jh) \operatorname{sinc}\left(\frac{t - jh}{h}\right), \tag{2}$$

where h is the sampling period and

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

- The sampling frequency is $\frac{1}{h}$, so the Nyquist rate condition for the Sampling Theorem is that $\frac{1}{h} > 2\sigma_{max}$, or $h < \frac{1}{2\sigma_{max}}$.
- At sampling intervals $h > \frac{1}{2\sigma_{max}}$, high frequencies are aliased to lower frequencies, creating distortion.
- CD quality audio samples at a rate of 44.1 kHz, which was chosen to capture frequencies up to 22.05 kHz, higher than most humans can hear.

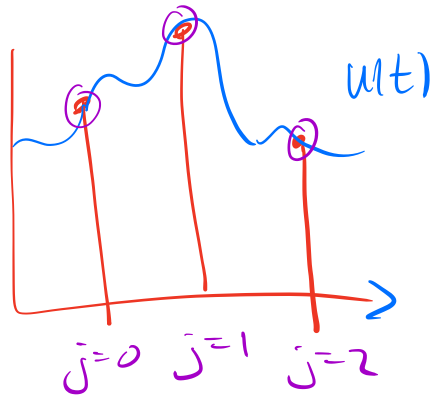
The Sampling Theorem (periodic version)

Let $u : \mathbb{R} \to \mathbb{R}$ be periodic with period 1, and assume u has no frequency larger than σ_{max}, where σ_{max} is a positive integer. This means that the signal u has the Fourier Series representation

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \, e^{2\pi i k t}. \tag{3}$$

Theorem 1. *Suppose that u is given by (3) and let $h = 1/n$ for $n \in \mathbb{N}$ with $\boxed{n > 2\sigma_{max}.}$ Assume also that n is odd. Then $u(t)$ can be reconstruted from its evenly spaced samples $u(jh)$ and furthermore we have*

$$u(t) = \sum_{j=0}^{n-1} u(jh) S\left(\frac{t - jh}{h}\right), \tag{4}$$



where $S(t)$ is given by

$$S(t) = \frac{\mathrm{sinc}(t)}{\mathrm{sinc}(ht)}.$$

Sinc kernel

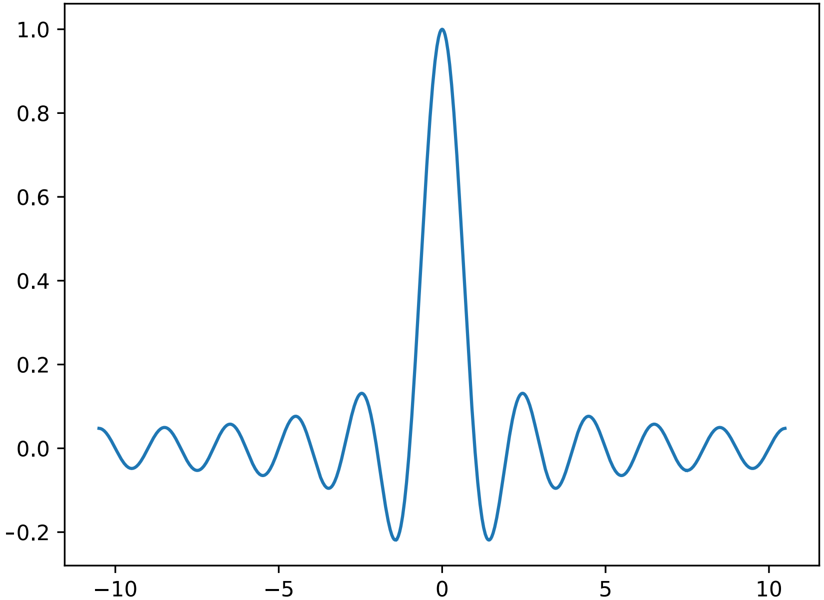


Figure 1: Depiction of the Sinc-like kernel $S(t) = \operatorname{sinc}(t)/\operatorname{sinc}(ht)$ for $n = 21$ and $h = 1/21$. The kernel is periodic with period $n = 21$.

Proof of Sampling Theorem

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \, e^{2\pi i k t}.$$

Define

$$u(t) = \sum_{j=0}^{n-1} u(jh) S\left(\frac{t-jh}{h}\right),$$

$$f(j) = u(jh) = u\left(\frac{j}{n}\right)$$

Since $j = 0, \dots, n-1$, and u is 1-periodic

we have $f : \mathbb{Z}_n \to \mathbb{R}$ is n-periodic.

So $f \in L^2(\mathbb{Z}_n)$.

$$f(j+n) = u\left(\frac{j+n}{n}\right)$$

$$= u\left(\frac{j}{n} + 1\right)$$

u is 1-periodic

$$= u\left(\frac{j}{n}\right)$$

$$= f(j)$$

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \, e^{2\pi i k t}.$$

$$f(j) = u\left(\frac{j}{n}\right) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k e^{2\pi i k j/n} = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \omega^{kj}$$

$$\omega = e^{2\pi i/n}$$

Recall $u_\ell(k) = \omega^{\ell k}$.

$$\langle f, u_\ell \rangle = \sum_{j=0}^{n-1} f(j)\, \omega^{-j\ell}$$

$$= \sum_{j=0}^{n-1} \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \omega^{j(k-\ell)}$$

$$\omega^n = 1$$

$$= \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \underbrace{\sum_{j=0}^{n-1} \omega^{j(k-\ell)}}$$

If $\boxed{k-\ell=nq}$

$$w^{j(k-\ell)}=w^{jnq}=1$$

$$\rightarrow \begin{cases} n, & k=\ell \pmod n \\ 0, & k\neq \ell \pmod n \end{cases}$$

$$\langle f, u_\ell\rangle = n\sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \underbrace{\delta_{\{k=\ell \bmod n\}}}_{=1 \text{ only when } k=\ell}.$$

$$\sigma_{max} < \frac{n}{2}$$

$$n > 2\sigma_{max} \iff n-1 \geq 2\sigma_{max}$$

$$\sigma_{max} \leq \frac{n-1}{2}$$

In this case $nc_\ell = \langle f, u_\ell\rangle$.

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k\, e^{2\pi i k t}. = \frac{1}{n} \sum_{k=-\sigma_{max}}^{\sigma_{max}} \langle f, u_k \rangle e^{2\pi i k t}$$

$\langle f, u_k \rangle = 0$

if $|\sigma_{max}| < k \le \frac{n-1}{2}$

$$= \frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle f, u_k \rangle e^{2\pi i k t}$$

$$= \frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{j=0}^{n-1} f(j) e^{-2\pi i j k/n} e^{2\pi i k t}$$

$$= \sum_{j=0}^{n-1} u(jh) \underbrace{\left(\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i k (t - jh)} \right)}_{\text{HW\#3}}$$

$$S(t) = \frac{\mathrm{sinc}(t)}{\mathrm{sinc}(ht)}.$$

Geometric Series $= S(t - jh)$.

$$\Phi = \frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} r^k, \quad r = e^{2\pi i (t - jh)}$$

$\blacksquare$

Discrete Cosine Transform

It is often useful in practical applications to avoid complex numbers and work with real-valued transformations. If $f \in L^2(\mathbb{Z}_n)$ is *real-valued* then the Fourier Inversion Theorem yields

$$\begin{aligned} f(k) &= \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)e^{2\pi i k\ell/n} \\ &= \frac{1}{n}\sum_{\ell=0}^{n-1}\sum_{j=0}^{n-1} f(j)e^{-2\pi i j\ell/n}e^{2\pi i k\ell/n} \\ &= \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\cos(2\pi j\ell/n)\right)\cos(2\pi k\ell/n) \\ &\qquad\qquad + \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\sin(2\pi j\ell/n)\right)\sin(2\pi k\ell/n). \end{aligned}$$

$$e^{-2\pi i j l/n} e^{2\pi i k l/n}$$

Euler's identity

$$= \left(\cos(2\pi j l/n) - i \sin(2\pi j l/n)\right)$$

$$\left(\cos(2\pi k l/n) + i \sin(2\pi k l/n)\right)$$

$$= \cos(2\pi j l/n)\cos(2\pi k l/n) + \sin(2\pi j l/n)\sin(2\pi k l/n)$$

$$+ i(\cdots\cdots)$$

Even/odd extensions

f $\quad\underbrace{ABCD}_{4}\,|\,\underbrace{CB}_{2}$

Let $f : \mathbb{Z}_n \to \mathbb{R}$. We define the even extension $f_e : \mathbb{Z}_{2(n-1)} \to \mathbb{R}$ by

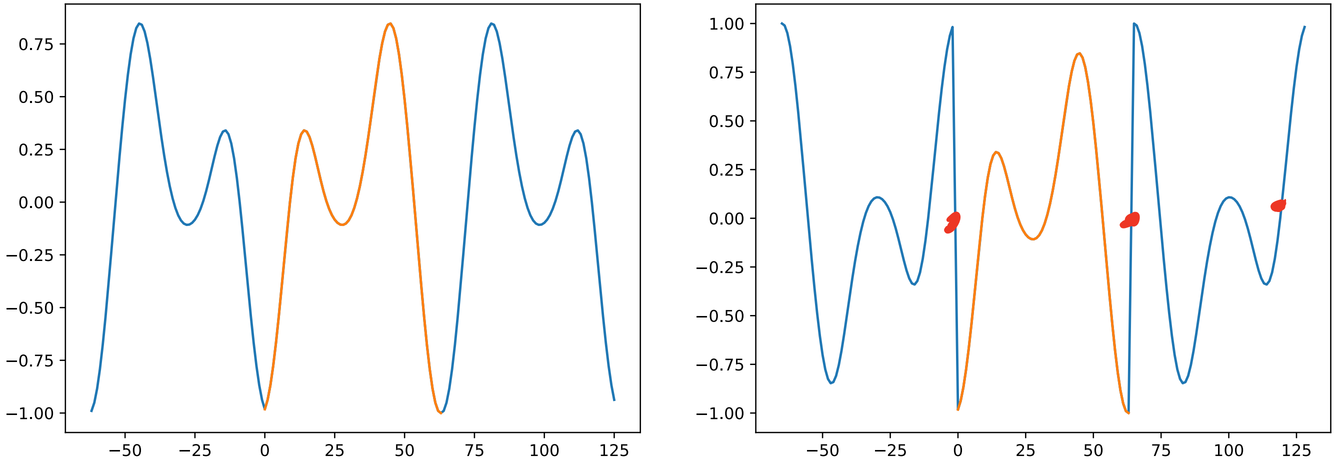
$$f_e(k) = \begin{cases} f(k), & \text{if } 0 \leq k \leq n-1, \\ f(2(n-1)-k), & \text{if } n \leq k \leq 2(n-1)-1. \end{cases} \tag{5}$$

$ABCD\,|\,0, -D, -C, -B, -A, 0$

The odd extension $f_o : \mathbb{Z}_{2(n+1)} \to \mathbb{R}$ is defined by

$$f_o(k) = \begin{cases} 0, & \text{if } k = 0 \\ f(k-1), & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k = n+1 \\ -f(2(n+1)-1-k), & \text{if } n+2 \leq k \leq 2(n+1)-1. \end{cases} \tag{6}$$

Even/odd extensions



(a) Even extension (b) Odd extension

Figure 2: Example of the even and odd extensions of a signal on $\mathbb{Z}_{64}$

Discete Cosine Transform

Recall

$$f(k) = \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\cos(2\pi j\ell/n)\right)\cos(2\pi k\ell/n)$$

A_ℓ

$$+\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\sin(2\pi j\ell/n)\right)\sin(2\pi k\ell/n).$$

B_ℓ

We now apply the representation formula above to the even extension f_e, taking $2(n-1)$ in place of n, to obtain the Discrete Cosine Transform

$$f(k) = \frac{1}{2(n-1)}(A_0 + (-1)^k A_{n-1}) + \frac{1}{n-1}\sum_{\ell=1}^{n-2} A_\ell \cos\left(\frac{\pi k\ell}{n-1}\right),$$

where

$$A_\ell = f(0) + (-1)^\ell f(n-1) + 2\sum_{k=1}^{n-2} f(k)\cos\left(\frac{\pi k\ell}{n-1}\right).$$

$$f_e(k) = \frac{1}{2(n-1)} \sum_{\ell=0}^{2(n-1)-1} A_\ell \cos\left(\frac{\cancel{2}\pi k \ell}{\cancel{2}(n-1)}\right)$$

$$+ \frac{1}{2(n-1)} \sum_{\ell=0}^{2(n-1)-1} B_\ell \sin\left(\frac{\pi k \ell}{n-1}\right)$$

$$A_\ell = \sum_{j=0}^{2(n-1)-1} f_e(j) \cos\left(\frac{\cancel{2}\pi j \ell}{\cancel{2}(n-1)}\right) = \cdots$$

$$B_\ell = \sum_{j=0}^{2(n-1)-1} f_e(j) \sin\left(\frac{\pi j \ell}{n-1}\right) \overset{\text{show}}{=} 0$$

$$f_e(k) = \begin{cases} f(k), & \text{if } 0 \le k \le n-1, \\ f(2(n-1)-k), & \text{if } n \le k \le 2(n-1)-1. \end{cases}$$

$$B_\ell = \sum_{j=0}^{2(n-1)-1} f_e(j)\sin\left(\frac{\pi j \ell}{n-1}\right)$$

$$= \sum_{j=0}^{n-1} f(j)\sin\left(\frac{\pi j \ell}{n-1}\right) + \sum_{j=n}^{2(n-1)-1} f(2(n-1)-j)\sin\left(\frac{\pi j \ell}{n-1}\right)$$

Change variables

$$k = 2(n-1) - j$$

$$= \sum_{j=0}^{n-1} f(j)\sin\left(\frac{\pi j \ell}{n-1}\right) + \sum_{k=1}^{n-2} f(k)\sin\left(\frac{\pi(2(n-1)-k)\ell}{n-1}\right)$$

$$= \sin\left(\frac{\pi 2(n-1)\ell}{n-1} - \frac{\pi k \ell}{n-1}\right)$$

$$= \sin\left(2\pi l - \frac{\pi k l}{n-1}\right)$$

$$= \sin\left(-\frac{\pi k l}{n-1}\right) \quad \leftarrow 2\pi\text{-periodic}$$

$$= -\sin\left(\frac{\pi k l}{n-1}\right)$$

$$= \sum_{j=0}^{n-1} f(j) \sin\left(\frac{\pi j l}{n-1}\right) - \sum_{k=1}^{n-2} f(k) \sin\left(\frac{\pi k l}{n-1}\right)$$

$$= f(n-1) \underbrace{\sin\left(\frac{\pi \cancel{(n-1)} l}{\cancel{n-1}}\right)}_{0} + f(0) \underbrace{\sin(0)}_{0}.$$

$$= 0$$

$$A_\ell = \sum_{j=0}^{2(n-1)-1} f_\ell(j) \cos\left(\frac{\pi j \ell}{n-1}\right)$$

$$= \sum_{j=0}^{n-1} f(j) \cos\left(\frac{\pi j \ell}{n-1}\right) + \sum_{j=n}^{2(n-1)-1} f(2(n-1)-j) \cos\left(\frac{\pi j \ell}{n-1}\right)$$

$$k = 2(n-1)-j$$

$$= \sum_{j=0}^{n-1} f(j) \cos\left(\frac{\pi j \ell}{n-1}\right) + \sum_{k=1}^{n-2} f(k) \cos\left(\frac{\pi (\cancel{2(n-1)}-k)\ell}{n-1}\right)$$

$$= \sum_{j=0}^{n-1} f(j) \cos\left(\frac{\pi j \ell}{n-1}\right) + \sum_{k=1}^{n-2} f(k) \cos\left(\frac{\pi k \ell}{n-1}\right)$$

$$A_\ell = f(0) + f(n-1)\underbrace{\cos(\pi\ell)}_{=(-1)^\ell} + 2\sum_{k=1}^{n-2} f(k)\cos\left(\frac{\pi k \ell}{n-1}\right)$$

Discrete Sine Transform

Using the odd extension we get the Discrete Sine Transform

$$f(k) = \frac{1}{n+1} \sum_{\ell=0}^{n-1} B_\ell \sin\left(\frac{\pi(k+1)(\ell+1)}{n+1}\right), \tag{7}$$

where

$$B_\ell = 2 \sum_{k=0}^{n-1} f(k) \sin\left(\frac{\pi(k+1)(\ell+1)}{n+1}\right),$$

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