Mathematics of Image and Data Analysis
Math 5467

Signal Denoising

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Last time

- Parseval and Convolution

Today

- Signal Denoising with Tikhonov regularization

Announcemnets

- No office hours this week (unless by appointment).
- Wed lecture over Zoom only

Recall

Definition 1. The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \sum_{k=0}^{n-1} f(k)\omega^{-k\ell} = \sum_{k=0}^{n-1} f(k)e^{-2\pi i k\ell/n},$$

where $\omega = e^{2\pi i/n}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$.

Definition 2 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{2\pi i k\ell/n}.$$

Lemma 3 (Convolution and the DFT)**.** *For $f, g \in L^2(\mathbb{Z}_n)$ we have*

$$\mathcal{D}(f * g) = \mathcal{D}f \cdot \mathcal{D}g. \tag{1}$$

Exercise on discrete derivatives

Exercise 4. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

(i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

$$\nabla^- f(k) = f(k) - f(k-1).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = g * f$ and use this with Lemma 3 to show that $\mathcal{D}(\nabla^- f)(k) = (1 - \omega^{-k})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

(ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

$$\nabla^+ f(k) = f(k+1) - f(k).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^+ f = g * f$ and use this with Lemma 3 to show that $\mathcal{D}(\nabla^+ f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

(iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

$$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i\sin(2\pi k/n)\mathcal{D}f(k).$$

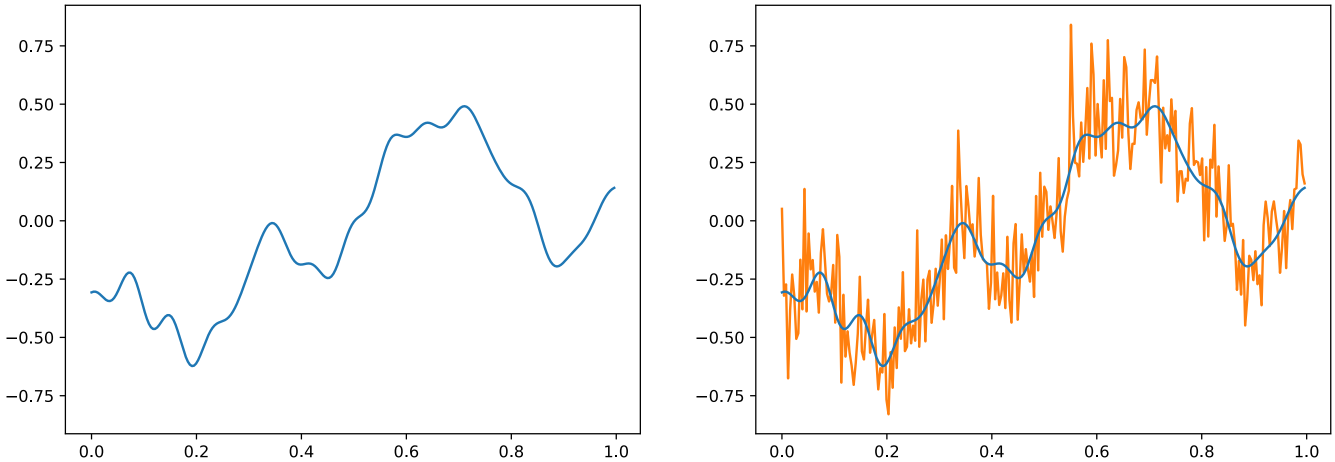
(iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

$$\Delta f(k) = \nabla^+\nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k). \qquad \triangle$$

Signal denoising



(a) Clean signal (b) Noisy signal

Figure 1: Example of a clean signal, and a noisy version, corrupted with additive Gaussian noise. The clean signal is superimposed over the noisy one in (b) for reference.

Sources of noise

A brief (and not exhaustive) list of sources of signal noise.

- Signal acquisition (microphone diaphragm, camera sensor).
- Thermal noise in electronic circuitry.
- Noise from signal compression artifacts.
- Corruption of wireless signals.

Tikhonov regularization

Let $f \in L^2(\mathbb{Z}_n)$ be the noisy signal. Tikhonov regularized denoising minimizes the energy $E : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$E(u) = \underbrace{\sum_{k=0}^{n-1} |u(k) - f(k)|^2}_{\text{Data Fidelity}} + \lambda \underbrace{\sum_{k=0}^{n-1} \overbrace{|u(k) - u(k-1)|}^{\nabla^- u(k)}{}^2}_{\text{Regularizer}}, \tag{2}$$

where $\lambda \geq 0$ is a parameter.

Main ideas:

- Data fidelity keeps the denoised signal close to the noisy signal f.
- Regularizer removes the noise.

Tikhonov regularization

$\|u\|^2 = \sum_{k=0}^{n-1} |u(k)|^2$

We recall the backward difference $\nabla^- : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ is defined by

$$\nabla^- u(k) = u(k) - u(k-1),$$

while the forward difference is $\nabla^+ u(k) = u(k+1) - u(k)$. The discrete Laplacian is

$$\Delta u = \nabla^+ \nabla^- u = \nabla^- \nabla^+ u.$$

In terms of this notation, the Tikhonov regularized denoising problem is

$$\min_{u \in L^2(\mathbb{Z}_n)} E(u) = \underbrace{\|u - f\|^2}_{\text{Fidelity}} + \lambda \underbrace{\|\nabla^- u\|^2}_{\text{Reg.}}. \tag{3}$$

Theorem 5. *Let $\lambda \geq 0$ and $f \in L^2(\mathbb{Z}_n)$. Then there exists a unique solution $u \in L^2(\mathbb{Z}_n)$ of the optimization problem (3). Furthermore, the minimizer u is also characterized as the unique solution of the Euler-Lagrange equation*

$$u - \lambda \Delta u = f. \tag{4}$$

$\nabla E = 0$

Properties of discrete derivatives

Before proving the theorem, we need to establish some basic properties of discrete derivatives. These are the discrete analogs of integration by parts formulas.

Proposition 6. *For all $u, v \in L^2(\mathbb{Z}_n)$ the following hold.*

(i) $\langle \nabla^- u, v \rangle = -\langle u, \nabla^+ v \rangle$

$\int u\,dv = \cancel{uv} - \int v\,du$

(ii) $\langle \nabla^+ u, v \rangle = -\langle u, \nabla^- v \rangle$

(iii) $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$ – Δ self adjoint

$\Delta \langle \nabla^- \nabla^+ u, v \rangle \overset{(i)}{=} -\langle \nabla^+ u, \nabla^+ v \rangle \overset{(ii)}{=} \langle u, \nabla^- \nabla^+ v \rangle = \langle u, \Delta v \rangle$

Remark 7. We can take $u = v$ to obtain

$u = v$

$$\|\nabla^+ u\|^2 = \|\nabla^- u\|^2 = -\langle \Delta u, u \rangle.$$

(i) $\langle \nabla^- u, v \rangle = \sum_{k=0}^{n-1} \nabla^- u(k) \overline{v(k)}$

$$= \sum_{k=0}^{n-1} \left(u(k) - u(k-1) \right) \bar{v}(k)$$

$$= \sum_{k=0}^{n-1} u(k) \bar{v}(k) - \underbrace{\sum_{k=0}^{n-1} u(k-1) \bar{v}(k)}_{= \sum_{j=-1}^{n-2} u(j) \bar{v}(j+1)} \quad j = k-1$$

$$= \sum_{k=0}^{n-1} u(k) \bar{v}(k) - \sum_{k=0}^{n-1} u(k) \bar{v}(k+1) .$$

$$= \sum_{k=0}^{n-1} u(k) \underbrace{\overline{\left(v(k) - v(k+1)\right)}}_{-\nabla^+ v(k)} = -\langle u, \nabla^+ v \rangle.$$

$\boxed{///}$

Proof of Theorem 5:

(4) $u - \lambda \Delta u = f.$

① Equation (4) has a unique solution u.

Uniqueness: Let u, v solve (4)

$$u - \lambda \Delta u = f$$
$$-\left(v - \lambda \Delta v = f\right)$$

Set $w = u - v$: $\quad w - \lambda \Delta w = 0$

Take inner product $\langle \cdot, \cdot \rangle$ of both sides with w

$$\langle w - \lambda \Delta w, w \rangle = 0$$

$$\|\nabla^+ u\|^2 = \|\nabla^- u\|^2 = -\langle \Delta u, u\rangle$$

$$\langle w, w\rangle - \lambda \langle \Delta w, w\rangle = 0$$

$$\|w\|^2 + \lambda \|\nabla^+ w\|^2 = 0$$

$\Rightarrow$ $w = 0 \Rightarrow u = v$ and sol^{n} is unique for $\lambda \geq 0$.

Existence follows from Rank-Nullity Theorem

$$u - \lambda \Delta u = f \iff \boxed{Ax = f}$$

② Let $u \in \ell^2(\mathbb{Z}_n)$ solve $u - \lambda \Delta u = f$

Claim: $E(u) \leq E(w) \quad \forall w \in \ell^2(\mathbb{Z}_n)$

Calculus of Variations: Fix $w \in \ell^2(\mathbb{Z}_n)$ and $v = w - u$, and set

$$e(t) = E(u + tv), \quad t \in \mathbb{R}$$

$$E(w) - E(u) = e(1) - e(0) = \underbrace{\int_0^1 e'(t)\,dt}_{\geq 0}$$

$$e'(t) = \frac{d}{dt} E(u + tv)$$

$\longrightarrow E(u) \leq E(w)$

$$= \underbrace{\frac{d}{dt} \|u + t\overset{-f}{v}\|^2}_{} + \underbrace{\lambda \frac{d}{dt} \|D^-(u+tv)\|^2}_{B}$$

$$= \frac{d}{dt} \langle u + t\overset{-f}{v}, u + t\overset{-f}{v} \rangle$$

$$= \frac{d}{dt} \Big(\langle u, u+tv \rangle + \langle tv, u+tv \rangle \Big)$$

$$= \frac{d}{dt} \Big(\langle u, u \rangle + t\langle u, v \rangle + t\langle v, u \rangle + t^2 \langle v, v \rangle \Big)$$

$$= \langle u, v \rangle + \langle v, u \rangle + 2t\|v\|^2$$

$$= \langle u, v \rangle + \overline{\langle u, v \rangle} + 2t\|v\|^2$$

$z = a + ib$

$\bar{z} = a - ib$

$z + \bar{z} = 2a$

$\overline{\langle u, v \rangle} = \langle v, u \rangle$

$$= 2\,\mathrm{Re}\langle \underset{u-f}{u}, v\rangle + 2t\|v\|^2$$

$$B = \frac{d}{dt}\langle \nabla^- u + t\nabla^- v, \nabla^- u + t\nabla^- v\rangle$$

$$= \frac{d}{dt}\left(\|\nabla^- u\|^2 + 2t\,\mathrm{Re}\langle \nabla^- u, \nabla^- v\rangle + t^2\|\nabla^- v\|^2 \right)$$

$$= 2\,\mathrm{Re}\langle \nabla^- u, \nabla^- v\rangle + 2t\|\nabla^- v\|^2$$

IBP

$$= -2\,\mathrm{Re}\langle \nabla^+\nabla^- u, v\rangle + 2t\|\nabla^- v\|^2$$

$$= -2\,\mathrm{Re}\langle \Delta u, v\rangle + 2t\|\nabla^- v\|^2$$

Maximum principle

Proposition 8 (Maximum Principle)**.** *Let $\lambda \geq 0$ and let $f \in L^2(\mathbb{Z}_n)$ be real-valued. Let $u \in L^2(\mathbb{Z}_n)$ be the denoised signal, i.e., the solution of*

$$u - \lambda \Delta u = f,$$

which is also real-valued. Then we have

$$\min_{\mathbb{Z}_n} f \leq u \leq \max_{\mathbb{Z}_n} f. \tag{5}$$

If u is any minimizer of E then

$$0 = e'(0) = \frac{d}{dt} E(u + tv) = 2\,\mathrm{Re}\langle u - \lambda \Delta u - f, v\rangle$$

since $e(0) \leq e(t)$

$\forall v \in L^2(\mathbb{Z}_n)$

$\Longrightarrow u - \lambda \Delta u = f.$ ∎

Proof continued:

First variation

$$e'(t) = \frac{d}{dt} E(u+tv)$$

$$= 2\,\mathrm{Re}\langle u - f, v\rangle + 2t\|v\|^2$$

$$- 2\lambda\,\mathrm{Re}\langle \Delta u, v\rangle + 2t\lambda\|\nabla v\|^2$$

$$= 2\,\mathrm{Re}\langle \underbrace{u - \lambda\Delta u - f}_{=0 \text{ since } u-\lambda\Delta u = f}, v\rangle + 2t\left(\|v\|^2 + \lambda\|\nabla v\|^2\right)$$

Hence $e'(t) \geq 0$, $t \geq 0$

$$\Longrightarrow E(u) \leq E(w)$$

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k).$$

How to solve the Euler-Lagrange equation?

We can use the DFT (and FFT) to solve the Tikhonov regularized denoising equation

$$\mathcal{D}(u - \lambda\Delta u) = \mathcal{D}f.$$

Indeed, the solution is given by

$$u = \mathcal{D}^{-1}(G_\lambda \cdot \mathcal{D}f). \tag{6}$$

where

$$G_\lambda(k) = \frac{1}{1 + 2\lambda - 2\lambda\cos(2\pi k/n)}.$$

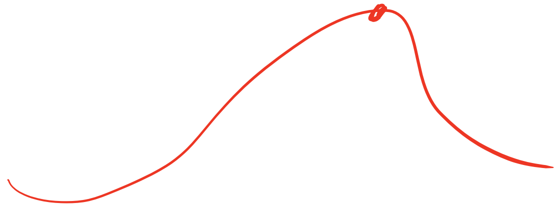
$$\mathcal{D}u - \lambda\mathcal{D}(\Delta u) = \mathcal{D}f$$

$$\mathcal{D}u(k) - 2\lambda((\cos(2\pi k/n) - 1)\mathcal{D}u(k) = \mathcal{D}f(k)$$

$$\mathcal{D}u(k)\left(1 + 2\lambda - 2\lambda\cos(2\pi k/n)\right) = \mathcal{D}f(k)$$

Maximum principle

Proposition 8 (Maximum Principle). *Let $\lambda \geq 0$ and let $f \in L^2(\mathbb{Z}_n)$ be real-valued. Let $u \in L^2(\mathbb{Z}_n)$ be the denoised signal, i.e., the solution of*



$$\boxed{u - \lambda \Delta u = f,}$$

which is also real-valued. Then we have

(5) $$\min_{\mathbb{Z}_n} f \leq u \leq \max_{\mathbb{Z}_n} f.$$

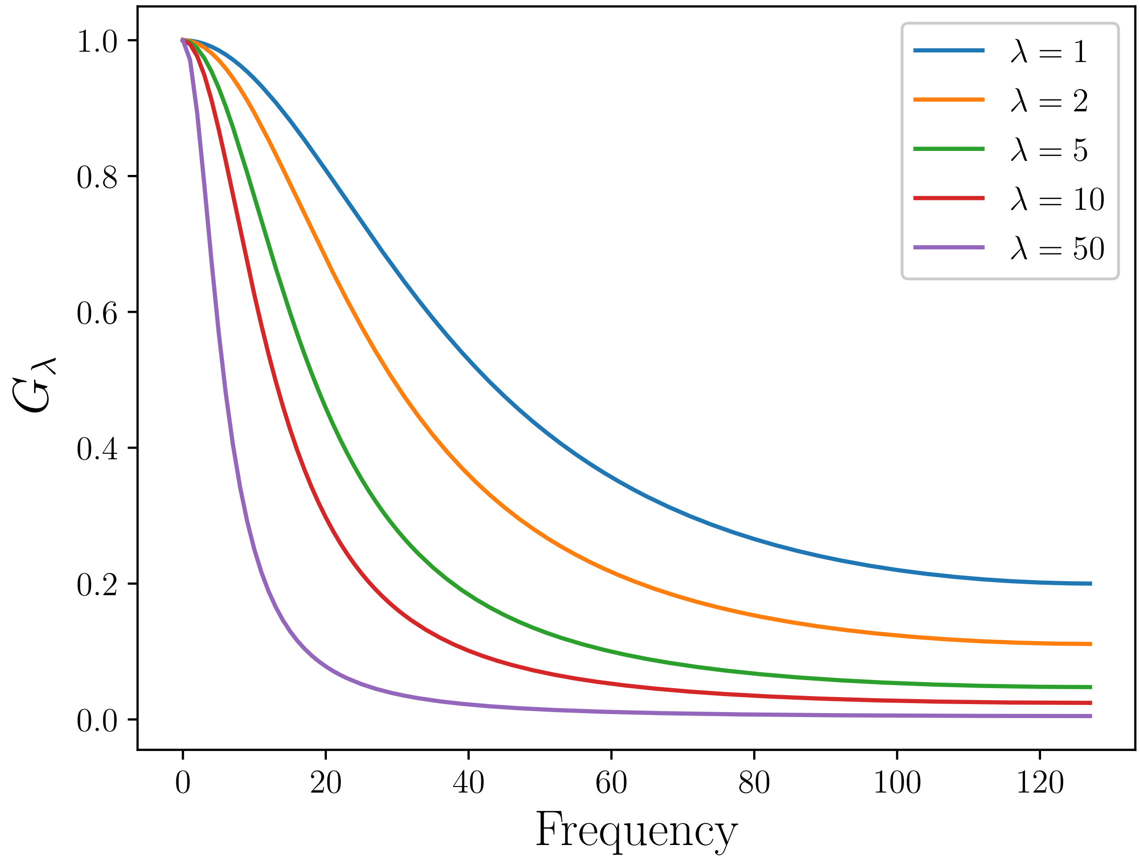
Proof: Let k be a maximum of u: $u(k) \geq u(j)$ $\forall j$. Then

$$\Delta u(k) = \underbrace{u(k+1)}_{\leq u(k)} - 2u(k) + \underbrace{u(k-1)}_{\leq u(k)} \leq 0$$

$$f(k) = u(k) - \lambda \Delta u(k) \geq u(k) = \max u.$$

Similar argument for $\min u \geq \min f$.

The low-pass filter G_λ



The filtering perspective

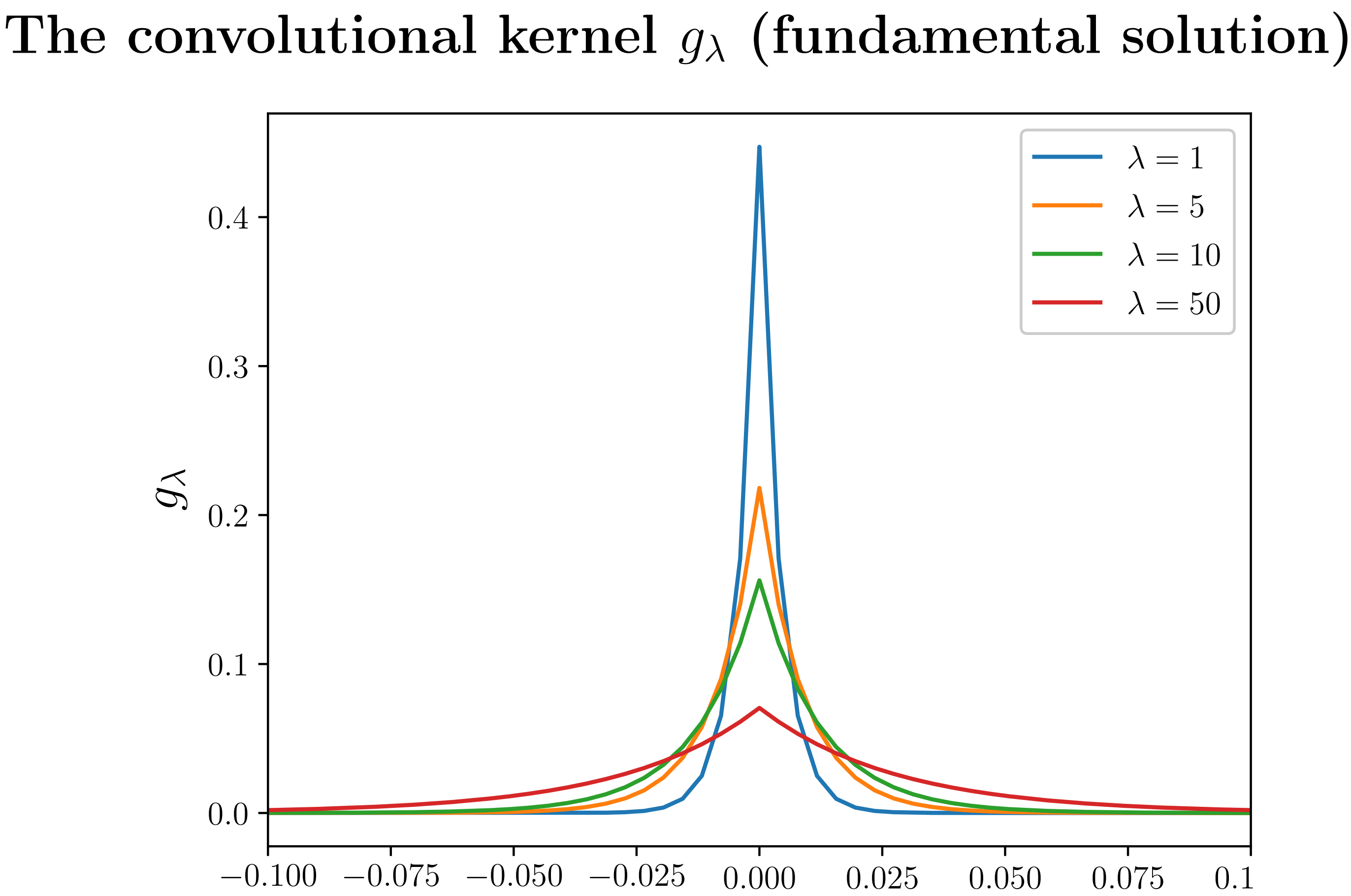
$u = D^{-1}(G_\lambda Df)$

Defining $g_\lambda = \mathcal{D}^{-1} G_\lambda$ we have

$$u = \mathcal{D}^{-1}(\mathcal{D} g_\lambda \cdot \mathcal{D} f) = g_\lambda * f.$$

The function g_λ is often called a *kernel*. The convolution to compute $u(k)$ should be thought of as a weighted average, weighted by g_λ, of the values of f nearby k. Indeed, we have

$$u(k) = (g_\lambda * f)(k) = \sum_{j=0}^{n-1} g_\lambda(j) f(k-j).$$



The convolutional kernel g_λ (fundamental solution)
g_λ
0.0
0.1
0.2
0.3
0.4
−0.100
−0.075
−0.050
−0.025
0.000
0.025
0.050
0.075
0.1
$\lambda = 1$
$\lambda = 5$
$\lambda = 10$
$\lambda = 50$

Tikhonov denoising (.ipynb)