Math 5490 – Homework 5: Due April 5 by 11:59pm

Instructions:

- Complete the problems below, and submit your solutions and Python code by uploading them to the Google form: https://forms.gle/5PELk1XogtFrDYFD6
- Submit all your Python code in a single .py file using the function templates given in each problem. I will import your functions from this file and test your code.
- If you use LaTeX to write up your solutions, upload them as a pdf file. Students who use LaTeX to write up their solutions will receive bonus points on the homework assignment (equivalent to 1/3 of a letter grade bump).
- If you choose to handwrite your solutions and scan them, please either use a real scanner, or use a smartphone app that allows scanning with you smartphone camera. It is not acceptable to submit photos of your solutions, as these can be hard to read.

Problems:

1. Consider diffusion with a source term $\mathbf{f} \in \mathbb{R}^m$:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - L_{\mathrm{rw}}^T \mathbf{x}_k + \mathbf{f}. \tag{1}$$

The interpretation is that we are adding or removing (based on the sign of $\mathbf{f}$) mass at each iteration. Throughout this problem you should assume the graph is connected.

(i) Show that

$$\mathbf{1}^T \mathbf{x}_{k+1} = \mathbf{1}^T \mathbf{x}_k + \mathbf{1}^T \mathbf{f}.$$

For the rest of the exercise assume that $\mathbf{1}^T \mathbf{f} = 0$, so that (1) conserves mass.

(ii) Show that the the mean zero condition $\mathbf{1}^T \mathbf{f} = 0$ implies that $\mathbf{f} \in \operatorname{img} L_{\mathrm{rw}}^T$, and so there exists $\mathbf{x} \in \mathbb{R}^m$ such that $L_{\mathrm{rw}}^T \mathbf{x} = \mathbf{f}$.

(iii) Set $\mathbf{u}_k = \mathbf{x}_k - \mathbf{x}$ and show that $\mathbf{u}_k$ satisfies the diffusion equation

$$\mathbf{u}_{k+1} = \mathbf{u}_k - L_{\mathrm{rw}}^T \mathbf{u}_k.$$

(iv) Suppose that $\mathbf{x}_0 = 0$. Use Theorem 8.27 to solve for $\mathbf{u}_k$, and then use that $\mathbf{x}_k = \mathbf{x} + \mathbf{u}_k$ to show that

$$\mathbf{x}_k = \mathbf{x} - D \sum_{i=1}^{m} (1 - \lambda_i)^k (\mathbf{x} \cdot \mathbf{p}_i) \mathbf{p}_i,$$

where $\mathbf{p}_i$ and λ_i are the eigenvectors and eigenvalues of the random walk graph Laplacian L_{rw}.

(v) Assume the graph is aperiodic (in addtion the the assumption that the graph is connected). Show that if $\mathbf{x} \cdot \mathbf{1} = 0$ then $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. What happens if $\mathbf{x} \cdot \mathbf{1} \neq 0$?

2. In this exercise you'll prove convergence of the PageRank iteration for a nonsymmetric graph. In particular, in this exercise we do *not* assume that $W = W^T$. Let us define $P = W^T D^{-1}$.

   (i) Show that P is nonexpansive in the 1 norm, that is, show that

   $$\|P\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1.$$

   (ii) Show that when $0 \leq \alpha < 1$ there is a unique solution $\mathbf{x}$ of the PageRank equation

   $$\mathbf{x} = (1-\alpha)\,\mathbf{v} + \alpha P\mathbf{x}. \tag{2}$$

   [Hint: Show that (2) is equivalent to $A\mathbf{x} = \mathbf{v}$ where

   $$A = (1-\alpha)^{-1}(I - \alpha P),$$

   and then use part (i) to show that $\ker A = \{0\}$.]

   (iii) Let $\mathbf{x}_k$ solve the PageRank iteration

   $$\mathbf{x}_{k+1} = (1-\alpha)\,\mathbf{v} + \alpha P\mathbf{x}_k, \tag{3}$$

   and let $\mathbf{x}$ be the PageRank vector, which satisfies (2). Show that

   $$\|\mathbf{x}_k - \mathbf{x}\|_1 \leq \alpha^k \|\mathbf{x}_0 - \mathbf{x}\|_1. \tag{4}$$

   [Hint: Subtract (2) from (3) and use part (i) to get that

   $$\|\mathbf{x}_{k+1} - \mathbf{x}\|_1 \leq \alpha \|\mathbf{x}_k - \mathbf{x}\|_1.$$

   Then complete the proof by induction.]

   (iv) Assume that the entries of $\mathbf{v}$ are nonnegative (i.e, $v_i \geq 0$). Show that the PageRank vector $\mathbf{x}$ also has nonnegative entries. [Hint: Choose $\mathbf{x}_0 = 0$ in the PageRank iteration (3), show that $\mathbf{x}_k$ has nonnegative entries, and use from part (iii) that $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$.]