Math 5587 – Homework 2 (Due Thursday Sept 15)

1. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous.

   (a) $u_t - u_{xx} + x = 0$.

   (b) $(x+y)u_{xy} + 2xu_y = x^2$.

   (c) $u_{xx} = e^u$.

   (d) $u_{txy} - u_{xx}u_{yy} + u_x = x^3$.

   (e) $u_t + u_{xxxxxx} - \sqrt{1+u^2} = 0$.

   (f) $u_t + u_x + u_y + u/xy = 0$.

2. Let u_* be a solution of the inhomogeneous linear equation $L[u_*] = g$. Show that **every** solution of $L[u] = g$ is of the form $u = u_* + v$, where v is a solution of the homogeneous linear equation $L[v] = 0$.

3. Find the solution to the initial value problem $u_t + u_x = 0$ satisfying $u(x,1) = x/(1+x^2)$.

4. Show that the only continuously differentiable solutions of $xu_x + yu_y = 0$ on the entire plane $\mathbb{R}^2$ are constant functions. [Hint: Show that for any fixed $(x,y) \in \mathbb{R}^2$, the function $g(t) = u(xt, yt)$ is constant in t.]

5. (a) Find a solution of $u_x u_y = 1$ on $\mathbb{R}^2$ of the form $u(x,y) = f(x) + g(y)$.

   (b) Find two different solutions of $u_x u_y = u$ in the domain $x \geq 0$ and $y \geq 0$ that satisfy $u(x,0) = 0$ and $u(0,y) = 0$ for all $x \geq 0$ and $y \geq 0$. [Hint: One is trivial. For the other, look for a solution in the separable form $u(x,y) = f(x)g(y)$.]

6. (a) Write down a formula for the general solution to the nonlinear PDE $u_t + u_x + u^2 = 0$.

   (b) Show that if the initial data $f(x) = u(x,0)$ is nonnegative and bounded $0 \leq f(x) \leq M$, then the solution exists for all $t > 0$, and $u(x,t) \to 0$ as $t \to \infty$.

   (c) On the other hand, if the initial data is negative at some x, show that the solution blows up in finite time: That is $\lim_{t \to \tau^-} u(y,t) = -\infty$ for some $\tau > 0$ and $y \in \mathbb{R}$.

   (d) Find a formula for the earliest blow-up time $\tau_* > 0$.