Math 5587 – Lecture 6

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We know how to solve the Cauchy problems for the wave and heat equations. For the wave equation we have d'Alembert's formula

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

where $f(x) = u(x,0)$ is the initial position and $g(x) = u_t(x,0)$ is the initial velocity of the string. For the heat equation we have the representation formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy,$$

where $f(x) = u(x,0)$ is the initial heat distribution. The Cauchy problem models the vibrations of an infinitely long string, and the diffusion of heat in an infinitely long insulated rod.

We consider here the heat and wave equations on the half-line $x \geq 0$. This models the vibrations of a semi-infinite string and diffusion within a semi-infinite rod. We must specify a boundary condition at $x = 0$, which will take to be Dirichlet $u(0,t) = 0$ or Neumann $u_x(0,t) = 0$ conditions.

1 Even and odd functions

Before addressing the half-line problems directly, we recall some properties of even and odd functions.

Definition 1. A function $f(x)$ is *even* if $f(x) = f(-x)$ for all x, and *odd* if $f(x) = -f(-x)$ for all x.

Lemma 1. *If f is an odd function that is continuous at $x = 0$, then $f(0) = 0$.*

Proof. Since f is odd

$$f(0) = \lim_{x\to 0^+} f(x) = -\lim_{x\to 0^+} f(-x) = -\lim_{x\to 0^-} f(x) = -f(0),$$

where the limits above hold due to the continuity of f at $x = 0$. Therefore $2f(0) = 0$, hence $f(0) = 0$. □

It follows from Lemma 1 that if $u(x,t)$ is an odd function of x for every t, then $u(0,t) = 0$. Hence u automatically satisfies the homogeneous Dirichlet boundary condition at $t = 0$.

Lemma 2. *If f is an even function that is differentiable at $x = 0$ then $f'(0) = 0$.*

Proof. Since $f(h) = f(-h)$ we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(-h)}{2h} = 0. \qquad \square$$

By Lemma 2, if $u(x,t)$ is an even function of x for every t, then $u_x(0,t) = 0$. Hence u automatically satisfies the homogeneous Neumann boundary conditions at $t = 0$.

Lemma 3. *Let $u(x,t)$ be a solution of the wave equation on $-\infty < x < \infty$, $t \geq 0$. If the initial data $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$ are even (respectively, odd) functions, then for all $t > 0$, $u(x,t)$ is an even (respectively, odd) function of x.*

The lemma shows that the solution of the wave equation on the entire real line preserves the evenness or oddness of the initial data.

Proof. Suppose f and g are even. By d'Alembert's formula

$$\begin{aligned} u(-x,t) &= \frac{1}{2}(f(-x+ct) + f(-x-ct)) + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(s)\,ds \\ &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(-s)\,ds \\ &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy = u(x,t), \end{aligned}$$

where we made the change of variables $y = -s$ in the final step. Therefore u is an even function of x.

The proof that u is odd when f and g are odd is similar. We leave it to Exercise 1. $\square$

Exercise 1. Complete the proof of Lemma 3 by showing that when f and g are odd, the solution u given by d'Alembert's formula is also an odd function of x.

We can prove the same result for the heat equation.

Lemma 4. *Let $u(x,t)$ be a solution of the heat equation on $-\infty < x < \infty$, $t \geq 0$. If the initial data $f(x) = u(x,0)$ is even (respectively, odd), then for all $t > 0$, $u(x,t)$ is an even (respectively, odd) function of x.*

Proof. Suppose f is an odd function. Then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy \\ &= -\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(-y)\,dy \\ &= -\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x+z)^2/4kt} f(z)\,dz \\ &= -u(-x,t). \end{aligned}$$

Therefore u is an odd function of x as well.

The proof that u is even when f is even is similar. $\square$

2 The odd extension: Dirichlet conditions

2.1 The heat equation

We are now equipped to solve the heat equation

$$u_t - ku_{xx} = 0 \quad \text{for} \quad x > 0 \text{ and } t > 0,$$

subject to the Dirichlet boundary condition $u(0,t) = 0$ and the initial condition $u(x,0) = f(x)$. The idea is to look instead for an *odd* solution of the heat equation on the *entire* real line satisfying $u(x,0) = f(x)$ for $x \geq 0$. Then by Lemma 1, u automatically satisfies the Dirichlet boundary condition $u(0,t) = 0$.

Keeping in mind Lemma 4, we define the *odd extension* of f by

$$f_{\text{odd}}(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ -f(-x), & \text{if } x < 0, \end{cases}$$

and solve the heat equation on the entire real line with initial conditions $u(x,0) = f_{\text{odd}}(x)$. The solution is

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) f_{\text{odd}}(y)\, dy,$$

where Φ is the fundamental solution of the heat equation. Since f is an odd function, Lemma 4 guarantees that u is an odd function of x for every t, hence $u(0,t) = 0$.

Let us simplify this formula. We have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} \Phi(x-y,t) f_{\text{odd}}(y)\, dy + \int_{0}^{\infty} \Phi(x-y,t) f_{\text{odd}}(y)\, dy \\ &= -\int_{-\infty}^{0} \Phi(x-y,t) f(-y)\, dy + \int_{0}^{\infty} \Phi(x-y,t) f(y)\, dy \\ &= -\int_{0}^{\infty} \Phi(x+y,t) f(y)\, dy + \int_{0}^{\infty} \Phi(x-y,t) f(y)\, dy \\ &= \int_{0}^{\infty} (\Phi(x-y,t) - \Phi(x+y,t)) f(y)\, dy. \end{aligned}$$

Therefore, the solution of the half-line Dirichlet problem for the heat equation is given by

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right) f(y)\, dy.} \tag{1}$$

Notice the formula only uses the values of $f(y)$ for $y \geq 0$.

2.2 The wave equation

We can use the same odd extension technique for the Dirichlet half-line problem for the wave equation. We are looking for a solution of

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad x > 0 \text{ and } t > 0,$$

subject to the Dirichlet boundary condition $u(0,t) = 0$, and the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. We again define the odd extensions f_{odd} and g_{odd} of f and g, and solve the wave equation on the entire real line with initial conditions $u(x,0) = f_{\text{odd}}(x)$ and $u_t(x,0) = g_{\text{odd}}(x)$. d'Alembert's formula gives

$$u(x,t) = \frac{1}{2}(f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{\text{odd}}(s)\,ds.$$

By Lemma 3, u is an odd function of x and so $u(0,t) = 0$. All that is left is to simplify this formula.

Recall that $x > 0$. We have two cases to consider. First, suppose that $x \geq ct$. Then $x + ct \geq 0$ and $x - ct \geq 0$ and so

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Second, suppose that $0 < x < ct$. Then $x + ct \geq 0$ and $x - ct < 0$. Therefore

$$f_{\text{odd}}(x+ct) = f(x+ct) \quad \text{and} \quad f_{\text{odd}}(x-ct) = -f(ct-x),$$

with similar formulas holding for g_{odd} and g. Therefore

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{x-ct}^{0} -g(-s)\,ds + \frac{1}{2c}\int_{0}^{x+ct} g(s)\,ds \\
&= \frac{1}{2}(f(x+ct) - f(ct-x)) - \frac{1}{2c}\int_{0}^{ct-x} g(s)\,ds + \frac{1}{2c}\int_{0}^{x+ct} g(s)\,ds \\
&= \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)\,ds.
\end{aligned}$$

Therefore, the solution of the half-line wave equation with Dirichlet boundary conditions is

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)\,ds, & \text{if } 0 < x < ct \\ \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, & \text{if } x \geq ct. \end{cases}} \tag{2}$$

When $0 < x < ct$, the portion of the initial condition propagating left reflects off of $x = 0$ and affects the solution at (x,t). This is reflected in the first formula for $0 < x < ct$. When $x > ct$, the reflections are not felt by the solution, since they propagate at speed at most c. This is why we obtain exactly d'Alembert's formula in the second case.

3 The even extension: Neumann conditions

3.1 The heat equation

We now consider the heat equation

$$u_t - ku_{xx} = 0 \quad \text{for} \quad x > 0 \text{ and } t > 0,$$

on the half-line subject to the Neumann boundary condition $u_x(0,t) = 0$ and the initial condition $u(x,0) = f(x)$. We define the even extension of f by

$$f_{\text{even}}(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0, \end{cases}$$

and we solve the heat equation on the entire real line with initial condition $u(x,0) = f_{\text{even}}(x)$. The solution is

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) f_{\text{even}}(y)\, dy. \tag{3}$$

Since f is an even function, Lemma 4 guarantees that u is an even function of x for every t, hence $u_x(0,t) = 0$.

As before, we can simplify the solution greatly to obtain

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right) f(y)\, dy.} \tag{4}$$

We leave the details to the reader.

Exercise 2. Verify Equation (4) by simplifying (3).

3.2 The wave equation

We finally consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad x > 0 \text{ and } t > 0,$$

on the half-line subject to the Neumann boundary condition $u_x(0,t) = 0$, and the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. We again define the even extensions f_{even} and g_{even} of f and g, and solve the wave equation on the entire real line with initial conditions $u(x,0) = f_{\text{even}}(x)$ and $u_t(x,0) = g_{\text{even}}(x)$. d'Alembert's formula gives

$$u(x,t) = \frac{1}{2}(f_{\text{even}}(x+ct) + f_{\text{even}}(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g_{\text{even}}(s)\, ds. \tag{5}$$

By Lemma 3, u is an even function of x and so $u_x(0,t) = 0$.

As before, we can simplify this formula to obtain

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c} \int_{ct-x}^{ct+x} g(s)\, ds, & \text{if } 0 < x < ct \\ \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{c} \int_0^{ct-x} g(s)\, ds + \frac{1}{2c} \int_{ct-x}^{ct+x} g(s)\, ds, & \text{if } x \geq ct. \end{cases}} \tag{6}$$

The details are left to the reader

Exercise 3. Verify that Equation (6) holds by simplifying (5).