Lecture 10 | Math 5588 Feb 21, 2017

# Fourier Transform on $\mathbb{R}^n$.

Recall: For $f:\mathbb{R}\longrightarrow\mathbb{R}$ the Fourier transform is

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx$$

and the inverse transform is

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{ikx}\,dx.$$

For a function $u:\mathbb{R}^2\longrightarrow\mathbb{R}$, we take the Fourier transform in each variable separately:

$$\hat{u}(k_1,k_2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x_1,x_2)\,e^{-ik_1x_1}\,dx_1\right)e^{-ik_2x_2}\,dx_2$$

The order in which we take the transforms does not matter and we get

$$\hat{u}(k_1, k_2) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} u(x_1, x_2)\, e^{-i(k_1 x_1 + k_2 x_2)}\, dx_1\, dx_2$$

Writing $k = (k_1, k_2)$ and $x = (x_1, x_2)$ we have

$$\hat{u}(k) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} u(x)\, e^{-ik\cdot x}\, dx$$

where $k \cdot x = k_1 x_1 + k_2 x_2$ is the dot product.

Extending this to $\mathbb{R}^n$, we have the Fourier transform of $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\boxed{\hat{u}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)\, e^{-ik\cdot x}\, dx}$$

The inverse Fourier transform is defined by

$$\boxed{u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(k) e^{ik\cdot x} \, dk}$$

A lot of properties are similar to 1-D transform.

① Linearity

② Shifts: $\mathcal{F}(f(x-a)) = e^{-ik\cdot a} \hat{f}(k)$

③ Scaling: $\mathcal{F}(f(ax)) = \frac{1}{|a|^n} \hat{f}\left(\frac{k}{a}\right)$ $\left[\begin{array}{l}\text{Note: } y = a\,dx \\ dy = |a|^n dx\end{array}\right]$

④ Partial derivatives: Note $\frac{\partial}{\partial x_j}(k\cdot x) = k_j$

By inverse transform formula

$$u_{x_j}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(k) \frac{\partial}{\partial x_j} e^{ik\cdot x} \, dk$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} ik_j \hat{u}(k) e^{ik\cdot x} \, dk$$

Therefore

$$\boxed{\mathcal{F}(u_{x_j}) = i k_j \hat{u}(k)}$$

Similarly, $\mathcal{F}(u_{x_j x_j}) = (ik_j)^2 \hat{u}(k) = -k_j^2 \hat{u}(k)$

and

$$\mathcal{F}(\Delta u) = \mathcal{F}\left(\sum_{j=1}^{n} u_{x_j x_j}\right)$$

$$(\text{linearity}) \quad = \sum_{j=1}^{n} \mathcal{F}(u_{x_j x_j})$$

$$= \sum_{j=1}^{n} -k_j^2 \hat{u}(k)$$

$$= -|k|^2 \hat{u}(k) \qquad \left[|k|^2 = \sum_{j=1}^{n} k_j^2\right]$$

Hence:

$$\boxed{\mathcal{F}(\Delta u) = -|k|^2 \hat{u}(k)}$$

# Fourier transform of Gaussian:

$$u(x) = e^{-|x|^2/2} = \exp\left(-\tfrac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)\right)$$

$$\hat{u}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ik\cdot x}\, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n\text{-times}} e^{-\frac{x_1^2}{2} - ik_1x_1} \cdots e^{-\frac{x_n^2}{2} - ik_nx_n}\, dx_1 \cdots dx_n$$

$$= \prod_{j=1}^{n} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2} e^{-ik_jx_j}\, dx_j \right]$$

$$= \prod_{j=1}^{n} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ik_jx}\, dx \right] \leftarrow \text{1-D Fourier transform}$$

$$= \prod_{j=1}^{n} e^{-k_j^2/2} = e^{-|k|^2/2}$$

Therefore if $u(x) = e^{-|x|^2/2}$ then

$$\boxed{\hat{u}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/2} e^{-ik\cdot x}\, dx = e^{-|k|^2/2}}$$

Plancherel: As before

$$\langle \hat{u}, \hat{v} \rangle = \int_{\mathbb{R}^n} \hat{u}(k)\, \overline{\hat{v}(k)}\, dk = \int_{\mathbb{R}^n} u(x)\, \overline{v(x)}\, dx = \langle u, v \rangle$$

So $\mathcal{F}: L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is a Hilbert space isomorphism.

Convolution: Define the convolution of $u: \mathbb{R}^n \longrightarrow \mathbb{R}$ and $v: \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\boxed{(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy}$$

$f = u$
$g = v$

As before

$$\mathcal{F}(u*v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (u*v)(x)\, e^{-ik\cdot x}\, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y)\, v(x-y)\, e^{-ik\cdot x}\, dy\, dx$$

Set $z = x-y$, $dz = dx$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y)\, v(z)\, e^{-ik\cdot(y+z)}\, dy\, dz$$

$$= (2\pi)^{n/2} \left[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(y)\, e^{-ik\cdot y}\, dy \right] \left[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(z)\, e^{-ik\cdot z}\, dz \right]$$

$$= (2\pi)^{n/2}\, \hat{u}(k)\, \hat{v}(k)$$

Hence

$$\boxed{\mathcal{F}(u*v) = (2\pi)^{n/2}\, \hat{u}\cdot\hat{v}}$$

Back to heat equation:

$$\begin{cases} u_t - \Delta u = 0 \quad , \quad t > 0 \, , \quad x \in \mathbb{R}^n \\ u(x,0) = f(x). \end{cases}$$

Take Fourier transform in spatial variable

$$\hat{u}(k,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x,t) e^{-ik\cdot x} \, dx$$

Then

$$\begin{cases} \frac{d}{dt} \hat{u}(k,t) = -|k|^2 \hat{u}(k,t) \\ \hat{u}(k,0) = \hat{f}(k) \end{cases}$$

Solution to this ODE is

$$\boxed{\hat{u}(k,t) = \hat{f}(k) e^{-|k|^2 t}}$$

By convolution property

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} f * \mathcal{F}^{-1}\left(e^{-|k|^2 t}\right)$$

Recall: $\mathcal{F}\left(e^{-|x|^2/2}\right) = e^{-|k|^2/2}$

and $\mathcal{F}\left(u\left(\frac{x}{a}\right)\right) = |a|^n \hat{u}(ak)$

Set $a = \sqrt{2t}$ to get

$$\mathcal{F}\left(e^{-\left|\frac{x}{\sqrt{2t}}\right|^2/2}\right) = (2t)^{n/2} e^{-|k|^2 t}$$

Therefore,

$$\mathcal{F}\left(\frac{1}{(2t)^{n/2}} e^{\frac{-|x|^2}{4t}}\right) = e^{-|k|^2 t}$$

and $\mathcal{F}^{-1}\left(e^{-|k|^2 t}\right) = \frac{1}{(2t)^{n/2}} e^{\frac{-|x|^2}{4t}}$

Finally we have

$$u(x,t) = f * \underbrace{\left( \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} \right)}_{\Phi(x,t)} = f * \Phi$$

we call $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}}$

the fundamental solution of the heat equation

and the solution of

$$u_t - \Delta u = 0 \quad , \qquad u(x,0) = f(x)$$

is given by

$$u(x,t) = f * \Phi$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{\frac{-|x-y|^2}{4t}} dy$$

# Heat Equation with forcing

$$\begin{cases} u_t - \Delta u = f, & t > 0, \quad x \in \mathbb{R}^n \\ u(x,0) = 0 \end{cases}$$

Here $f = f(x,t)$, $u = u(x,t)$. Take Fourier transform in $x$

$$\hat{u}(k,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x,t)\, e^{-ik\cdot x}\, dx, \quad \hat{f}(k,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x,t)\, e^{-ik\cdot x}\, dx$$

Then

$$\begin{cases} \frac{d}{dt}\hat{u}(k,t) + |k|^2 \hat{u}(k,t) = \hat{f}(k,t) \\ \hat{u}(k,0) = 0 \end{cases}$$

Multiply both sides by integrating factor $e^{|k|^2 t}$

$$e^{|k|^2 t} \frac{d}{dt}\hat{u}(k,t) + |k|^2 e^{|k|^2 t} \hat{u}(k,t) = e^{|k|^2 t} \hat{f}(k,t)$$

$$\frac{d}{dt}\left(e^{|k|^2 t} \hat{u}(k,t)\right) = e^{|k|^2 t} \hat{f}(k,t)$$

Integrating and using $\hat{u}(k,0)=0$ we have

$$e^{|k|^2 t}\,\hat{u}(k,t) = \int_0^t e^{|k|^2 s}\,\hat{f}(k,s)\,ds$$

or

$$\hat{u}(k,t) = \int_0^t e^{-|k|^2(t-s)}\,\hat{f}(k,s)\,ds$$

Therefore

$$u(x,t) = \check{F}^{-1}\left(\int_0^t e^{-|k|^2(t-s)}\,\hat{f}(k,s)\,ds\right)$$

$$= \int_0^t \check{F}^{-1}\left(e^{-|k|^2(t-s)}\,\hat{f}(k,s)\right)ds$$

$$= \int_0^t \Big(f(\cdot,s) * \Phi(\cdot,\, t-s)\Big)(x)\,ds$$

$$= \int_0^t \int_{\mathbb{R}^n} f(y,s)\,\Phi(x-y,\,t-s)\,dy\,ds$$

We can now solve

$$\begin{cases} u_t - \Delta u = f & , \text{ in } t>0, \quad x \in \mathbb{R}^n \\ u(x,0) = g \end{cases}$$

by summing the two previous solutions [by linearity].

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} g(y)\,\Phi(x-y,t)\,dy + \int_0^t \int_{\mathbb{R}^n} f(y,s)\,\Phi(x-y,t-s)\,dy\,ds}$$

## Bessel Potentials

Consider the PDE

$$u - \Delta u = f \quad \text{in} \quad \mathbb{R}^n$$

Take the Fourier transform on both sides to get

$$\hat{u}(k) + |k|^2 \hat{u}(k) = \hat{f}(k) \quad , \quad k \in \mathbb{R}^n$$

Therefore

$$\hat{u}(k) = \frac{\hat{f}(k)}{1+|k|^2}$$

and

$$u(x) = \frac{1}{(2\pi)^{n/2}} (f * B)(x)$$

where $\hat{B}(k) = \dfrac{1}{1+|k|^2} = \displaystyle\int_0^\infty e^{-t(1+|k|^2)}\, dt$

neat trick, just compute integral on RHS.

Hence

$$B(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{B}(k)\, e^{ik\cdot x}\, dk$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_0^\infty e^{-t(1+|k|^2)} e^{ik\cdot x}\, dt\, dk$$

$$= \int_0^\infty e^{-t} \left[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t|k|^2} e^{ik\cdot x}\, dk \right] dt$$

$$= \int_0^\infty e^{-t} F^{-1}\left(e^{-t|k|^2}\right) dt$$ this was worked out earlier in notes

$$= \int_0^\infty e^{-t}\left(\frac{1}{(2t)^{n/2}} e^{\frac{-|x|^2}{4t}}\right) dt$$

$$= \int_0^\infty \frac{1}{(2t)^{n/2}} e^{-t-\frac{|x|^2}{4t}} dt$$

Hence $$\boxed{B(x) = \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t}}}{(2t)^{n/2}} dt}$$

$B$ is called a Bessel potential.

Finally, the solution of $u - \Delta u = f$ in $\mathbb{R}^n$ is

$$\boxed{\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}}(f * B)(x) \\ &= \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} f(y) \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{n/2}} dy\, dt \end{aligned}}$$