Lecture 11 | Math 5588 Feb 23, 2017

The Maximum Principle

The maximum principle is a very powerful technique for linear and non-linear PDE's of first and second order (elliptic) based on the simple idea that at a maximum a function's derivative vanishes, and its pure second derivatives are ≤ 0.

For instance, if $f: \mathbb{R} \to \mathbb{R}$ has a max at $x_0 \in \mathbb{R}$ then

$$\boxed{\begin{aligned} &f'(x_0) = 0 \\ \text{and} \quad &f''(x_0) \leq 0 \end{aligned}}$$

If $u: \mathbb{R}^n \to \mathbb{R}$ has a max at $x_0 \in \mathbb{R}^n$

then

$$\frac{d}{dt}\Big|_{t=0} u(x_0+tv) = 0 \qquad \text{for all } v \in \mathbb{R}^n$$

and $$\frac{d^2}{dt^2}\Big|_{t=0} u(x_0+tv) \leq 0 \qquad \text{for all } v \in \mathbb{R}^n$$

<u>Recall</u>

$$\frac{d}{dt}\Big|_{t=0} u(x_0+tv) = \nabla u(x_0) \cdot v = \sum_{i=1}^{n} u_{x_i}(x_0) v_i$$

and $$\frac{d^2}{dt^2}\Big|_{t=0} u(x_0+tv) = \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_j}(x_0) v_i v_j$$

<u>Therefore</u>, when $u: \mathbb{R}^n \longrightarrow \mathbb{R}$ has a maximum at $x_0 \in \mathbb{R}^n$ then

$$\boxed{\begin{array}{l} \nabla u(x_0) = 0, \quad \text{and} \\ \text{for all } v \in \mathbb{R}^n \quad \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_j}(x_0) v_i v_j \leq 0 \end{array}}$$

These two simple observations are actually very powerful when used cleverly.

First, some notation. The Hessian of u is the matrix

$$\nabla^2 u(x_0) = \begin{bmatrix} u_{x_1x_1}(x_0) & \cdots & u_{x_1x_n}(x_0) \\ \vdots & \ddots & \vdots \\ u_{x_nx_1}(x_0) & \cdots & u_{x_nx_n}(x_0) \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Definition: We say that $\nabla^2 u(x_0)$ is negative definite, written $\nabla^2 u(x_0) \le 0$, if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_ix_j}(x_0)\, v_i v_j \le 0 \qquad \text{for all } v \in \mathbb{R}^n$$

We say $\nabla^2 u(x_0)$ is positive definite, written $\nabla^2 u(x_0) \ge 0$, if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_ix_j}(x_0)\, v_i v_j \ge 0 \qquad \text{for all } v \in \mathbb{R}^n$$

Since $u_{x_i x_j} = u_{x_j x_i}$, D^2u is a symmetric matrix. ($D^2u^T = D^2u$).

Def: For symmetric matrices $A, B \in \mathbb{R}^{n \times n}$ we write $\boxed{A \leq B}$ whenever

$B - A \geq 0$, or $B - A$ positive definite.

This defines a partial order on symmetric matrices. It is partial because given A, B, it could be the case that neither $A \geq B$ nor $A \leq B$. The partial order is transitive, i.e,

$$A \leq B \text{ and } B \leq C \implies A \leq C.$$

We will use a slightly different version of necc. cond. for max.

Let $u, v : \mathbb{R}^n \to \mathbb{R}$. If $u - v$ has a maximum at x_0 then

$$D(u-v) = 0 \quad \text{at } x_0 .$$

and $D^2(u-v) \le 0$ (Hessian of $u-v$ is negative def.)

In other words, if $u-v$ has a max at $x_0 \in \mathbb{R}^n$ then

$$\boxed{\begin{aligned} Du(x_0) &= Dv(x_0) \\ \text{and} \quad D^2u(x_0) &\le D^2v(x_0) \end{aligned}}$$

We aim to use the maximum principle to prove uniqueness for nonlinear PDE of the form

$$\cancel{F(x, u, Du)}$$

$$\begin{cases} H(x,u,Du,D^2u) = 0 &, \text{ in } U \\ u = g &, \text{ on } \partial U \end{cases}$$

where $U \subset \mathbb{R}^n$ is open and bounded.

In particular, we will prove a <u>comparison principle</u>, which says that if

$$\begin{cases} H(x,u,Du,D^2u) \le H(x,v,Dv,D^2v) & \text{in } U \\ \text{and} \quad u \le v & \text{on } \partial U \end{cases}$$

then $\boxed{u \le v \text{ everywhere in } U}$

Basically, if there is an ordering in the PDE and B.C., then the solutions are ordered as well. <u>Uniqueness</u> follows directly from the <u>comparison principle</u>.

So, what conditions on H do we need to assume to prove a comparison principle?

Let us assume that

$$(*)\quad \begin{cases} H(x,u,Du,D^2u) < H(x,v,Dv,D^2v) & \text{in } U \\ \text{and} \quad u \le v & \text{on } \partial U. \end{cases}$$

The strict inequality is important and we'll return to this later. We want to prove $u \le v$ in U.

Assume to the contrary that for some $x \in U$, $u(x) > v(x)$. Let $x_0 \in U$ be a point at which $u - v$ attains its maximum. Since $u(x_0) > v(x_0)$ and $u \le v$ on ∂U, we know $x_0 \notin \partial U$.

By the necessary conditions for a max

$$u(x_0) > v(x_0)$$

$$\nabla u(x_0) = \nabla v(x_0) =: p$$

and $\quad \nabla^2 u(x_0) \leq \nabla^2 v(x_0)$

Plugging this into ($\#$) we have

$$0 < H(x_0, v(x_0), \nabla v(x_0), \nabla^2 v(x_0))$$
$$- H(x_0, u(x_0), \nabla u(x_0), \nabla^2 u(x_0))$$

$$= H(x_0, v(x_0), p, \nabla^2 v(x_0)) - H(x_0, u(x_0), p, \nabla^2 u(x_0))$$

If the RHS is ≤ 0 then we obtain a contradiction. This is true when

$$r \leq s \implies H(x, r, p, A) \leq H(x, s, p, A) \qquad (1)$$

and

$$A \leq B \implies H(x, r, p, B) \leq H(x, r, p, A) \qquad (2)$$

<u>Def</u>: We say H is <u>elliptic</u> when (1) and (2) are satisfied.

We can state the following theorem

<u>Theorem</u> If H is <u>elliptic</u> and

$$H(x,u,Du,D^2u) < H(x,v,Dv,D^2v) \quad \text{in } U$$

then $u \le v$ on $\partial U \implies u \le v$ in U.

<u>Example</u>: The standard example of an elliptic PDE is Poisson's equation

$$-\Delta u = f$$

Here $\quad H(x,u,Du,D^2u) = -\Delta u - f(x)$.

In other words
$$H(x,z,p,A) = -\sum_{i=1}^{n} a_{ii} - f(x)$$

when $\quad A = (a_{ij})_{i,j=1}^{n}$

To see that H is elliptic, let $A \leq B$

this means $B - A \geq 0$ or

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (b_{ij} - a_{ij}) v_i v_j \geq 0$$

for all $v \in \mathbb{R}^n$ when $B = (b_{ij})_{i,j=1}^{n}$

Take $v = e_k = (0, 0, \dots, 0, 1, 0, \dots, 0)$
$\uparrow_{k^{th}}$ entry

to get $b_{kk} - a_{kk} \geq 0$ or $b_{kk} \geq a_{kk}$

Therefore

$$\begin{aligned} H(x, z, p, A) &= -\sum_{i=1}^{n} a_{ii} - f(x) \\ &\geq -\sum_{i=1}^{n} b_{ii} - f(x) = H(x, z, p, B) \end{aligned}$$

and so H is elliptic.

Other elliptic PDEs include

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = 0$$

provided $(a_{ij})_{i,j=1}^{n} = A \geq 0$ and $c \geq 0$

the Monge-Ampère equation

$$-\det(D^2 u) = f$$

provided u is <u>convex</u>,

the minimal surface equation

$$-\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

In general, any Euler-Lagrange equation

$$-\operatorname{div}\left(D_p L(x, Du)\right) = 0$$

is elliptic

where L is convex in p.

Some of these will be on homework in coming week (proving ellipticity)

Also very important is that every first order equation

$$H(x, \nabla u) = 0$$

is elliptic in our definition

[Sometimes our definition is called degenerate ellipticity]

For our purposes, elliptic basically means the maximum principle holds.

We really want to prove a comparison principle *without* the strict inequality, so we *want* to show that for H elliptic

$$(*)\quad \begin{cases} H(x,u,Du,D^2u) \le H(x,v,Dv,D^2v) & \text{in } U \\ u \le v & \text{on } \partial U \end{cases}$$

implies $u \le v$ in U.

The trick is to find a way to perturb u or v so that the inequality in $(*)$ is *strict*, and then use previous theorem.

This is *not* always possible and depends on the structure of H.

We give a few special cases below.

__Theorem__ : If $H(x, Du, D^2u)$ is elliptic

and

$$u + H(x, Du, D^2u) \leq v + H(x, Dv, D^2v) \text{ in } U$$

then $u \leq v$ on $\partial U \implies u \leq v$ in U.

__Proof__ : For $\varepsilon > 0$ set $u_\varepsilon = u - \varepsilon$

Then $Du_\varepsilon = Du$ and $D^2 u_\varepsilon = D^2 u$.

Therefore

$$\begin{aligned} u_\varepsilon + H(x, Du_\varepsilon, D^2u_\varepsilon) &= u - \varepsilon + H(x, Du, D^2u) \\ &\leq v - \varepsilon + H(x, Dv, D^2v) \\ &< v + H(x, Dv, D^2v) \end{aligned}$$

and so

$$u_\varepsilon + H(x, Du_\varepsilon, D^2u_\varepsilon) < v + H(x, Dv, D^2v) \text{ in } U$$

By the comparison principle with

$$\tilde{H}(x, z, p, A) = z + H(x, p, A)$$

which is elliptic, we have

$$u_\varepsilon^* \leq v \quad \text{in } U$$

(note we used $u_\varepsilon \leq v$ on ∂U)

Send $\varepsilon \to 0^+$ to get $u \leq v$ in U $\boxed{}$

<u>Theorem</u>: If $\lambda \geq 0$ and

$$\lambda u - \Delta u \leq \lambda v - \Delta v \quad \text{in } U$$

then $u \leq v$ on $\partial U \implies u \leq v$ in U.

<u>Proof</u>: Let $\varepsilon > 0$ and define

$$u_\varepsilon = u + \frac{\varepsilon}{2n}|x|^2 - C\varepsilon$$

where $C > 0$ is chosen large enough so that

$$\frac{\varepsilon}{2n}|x|^2 - C\varepsilon \leq 0 \quad \text{in } U$$

In other words

$$|x|^2 \leq 2Cn \quad \text{for all } x \in U$$

This is possible because U is bounded.

Hence $\quad u_\varepsilon \leq u \quad$ and

$$\Delta u_\varepsilon = \Delta u + \varepsilon$$

So

$$\begin{aligned} \lambda u_\varepsilon - \Delta u_\varepsilon &= \lambda u_\varepsilon - \Delta u - \varepsilon \\ &\leq \lambda u - \Delta u - \varepsilon \\ &\leq \lambda v - \Delta v - \varepsilon \quad \text{in } U \end{aligned}$$

It follows that

$$\lambda u_\varepsilon - \Delta u_\varepsilon < \lambda v - \Delta v, \quad \text{in } U$$

Since $\quad H(x, u, Du, D^2u) = \lambda u - \Delta u \quad$ is <u>elliptic</u>

for $\lambda \geq 0$, we have $u_\varepsilon \leq v$ in U

Sending $\varepsilon \to 0^+$ we get $u \leq v$ $\boxtimes$

Next time we will consider time evolution equations

$$u_t + H(x, Du, D^2u) = 0$$