Maximum principle II: 3/7/2017 Math 5588

We now consider equations involving time

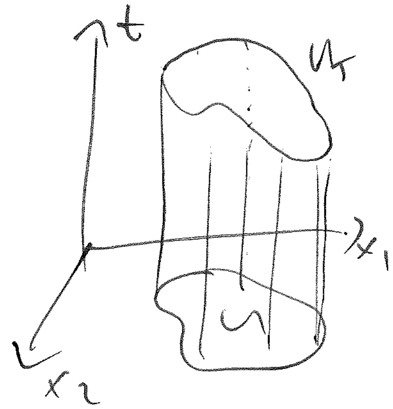
$$u_t + H(x, Du, D^2u) = 0 \qquad (*)$$

When H is elliptic, $(*)$ is called

<u>parabolic</u> (parabolic)

Let $U \subseteq \mathbb{R}^n$ be <u>open</u> and <u>bounded</u>.

We pose $(*)$ on the domain



$$U_T := U \times (0, T]$$

which is a <u>parabolic cylinder</u>

The <u>parabolic boundary</u> of U_T is

$$\Gamma_T = \overline{U_T} \setminus U_T$$

Γ_T = "sides and base of parabolic cylinder"

<u>Note</u>: Γ_T does <u>not</u> contain boundary when $t = T$

We can also write

$$\Gamma_T = \underbrace{\left(\bar{U} \times \{t=0\}\right)}_{\text{base}} \cup \underbrace{\left(\partial U \times (0,T)\right)}_{\text{sides}}$$

<u>Theorem</u>: (Comparison)

If $u(x,t)$ and $v(x,t)$ satisfy

$$\begin{cases} u_t + H(x, Du, D^2u) \leq v_t + H(x, Du, D^2u) & \text{in } U_T \\ u \leq v & \text{on } \Gamma_T \end{cases}$$

and H is <u>elliptic</u> then

$$\boxed{u \leq v \quad \text{in } U_T}$$

Proof: We will show that for every $\varepsilon > 0$

$$u_\varepsilon \le v \quad \text{in } U_T$$

where $$u_\varepsilon(x,t) = u(x,t) - \varepsilon t$$

Note $$(u_\varepsilon)_t = u_t - \varepsilon, \quad \nabla u_\varepsilon = \nabla u, \quad \nabla^2 u_\varepsilon = \nabla^2 u.$$

Thus

$$(*) \begin{cases} (u_\varepsilon)_t + H(x, \nabla u_\varepsilon, \nabla^2 u_\varepsilon) < v_t + H(x, \nabla v, \nabla^2 v), & \text{in } U_T \\ u_\varepsilon \le v, & \text{on } \Gamma_T \end{cases}$$

Assume to the contrary that

$$\max_{\overline{U_T}} (u_\varepsilon - v) > 0$$

that is $u_\varepsilon(x,t) > v(x,t)$ for some $(x,t) \in \overline{U_T}$

Then $u_\varepsilon - v$ attains its max over

$\overline{U_T}$ at some $(x_0, t_0) \in \overline{U_T}$.

Since $u_\varepsilon(x_0, t_0) - v(x_0, t_0) > 0$

and $u_\varepsilon \leq v$ on Γ_T

we have $(x_0, t_0) \in U_T$

Hence $\displaystyle \frac{\partial}{\partial t}\Big|_{t=t_0} (u_\varepsilon - v) \geq 0$ (since we might have $t_0 = T$)

and
$$\begin{cases} \nabla u_\varepsilon(x_0, t_0) = \nabla v(x_0, t_0) \\ \nabla^2 u_\varepsilon(x_0, t_0) \leq \nabla^2 v(x_0, t_0) \end{cases}$$

By ellipticity of H

$(u_\varepsilon)_t(x_0, t_0) + H(x_0, \nabla u_\varepsilon(x_0, t_0), \nabla^2 v(x_0, t_0)) \geq \cdots$

$$\dots\ v_t(x_0,t_0) + H(x_0, \nabla v(x_0,t_0), D^2 v(x_0,t_0))\Big)$$

which contradicts (*). Therefore

$$u_\varepsilon \le v \quad \text{on } U_T$$

for all $\varepsilon > 0$. Send $\varepsilon \to 0$ to get

$$u \le v \quad \text{on } U_T \qquad \blacksquare$$

<u>Some Applications of Maximum Principle</u>

Aside from proving uniqueness for solutions of <u>non-linear</u> PDEs, the maximum principle (or comparison principle) is very useful for deducing <u>properties</u> of solutions.

<u>Example</u>: (Decay of solutions of heat equation)
Let $U \subseteq \mathbb{R}^n$ be open and bounded and assume $u(x,t)$ satisfies

$$(H)\quad \begin{cases} u_t - \Delta u = 0, & \text{in } \mathcal{U}\times(0,\infty) \\ u = 0, & \text{on } \partial\mathcal{U}\times(0,\infty) \\ u(x,0) = f(x) \end{cases}$$

The Dirichlet condition $u=0$ on $\partial\mathcal{U}\times(0,\infty)$ means that the boundary $\partial\mathcal{U}$ is fixed at temperature $u=0$.

We expect that as $t\to\infty$

$$u \longrightarrow 0.$$

How fast does $u\longrightarrow 0$?

Recall the fundamental solution of the heat equation $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$

satisfies $\Phi_t - \Delta\Phi = 0$ for $t>0$.

Set $V(x,t) = \Phi(x, t+1)$

Since $U \subseteq \mathbb{R}^n$ is bounded, there exists $M \geq 0$ s.t.

$$|x| \leq M \text{ for all } x \in U.$$

Note:

$$V(x,0) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}} \geq \frac{e^{-\frac{M^2}{4}}}{(4\pi)^{n/2}}$$

Choose $C > 0$ so that

$$C V(x,0) \geq f(x) = u(x,0).$$

That is, select C so that

$$C \frac{e^{-\frac{M^2}{4}}}{(4\pi)^{n/2}} \geq \max_{x \in U} |f(x)|$$

or

$$C = (4\pi)^{n/2} e^{\frac{M^2}{4}} \max_{x \in U} |f(x)|$$

By comparison principle $u(x,t) \leq C\,v(x,t)$

which gives

$$u(x,t) \leq (4\pi)^{n/2} e^{\frac{M^2}{4}} \max_{x \in U} |f(x)| \cdot \frac{1}{(4\pi(t+1))^{n/2}} e^{\frac{-|x|^2}{4(t+1)}}$$

$$\leq \frac{e^{\frac{M^2}{4}} \max_{x \in U} |f(x)|}{(t+1)^{n/2}}$$

we can simarily show that

$$u(x,t) \geq \frac{-e^{\frac{M^2}{4}} \max_{x \in U} |f(x)|}{(t+1)^{n/2}}$$

Therefore

$$|u(x,t)| \le \frac{e^{\frac{M^2}{4}} \max_{x \in U} |f(x)|}{(t+1)^{n/2}}$$

Hence $u \longrightarrow 0$ faster than $\dfrac{C}{t^{n/2}}$ //

Example 2: (p Laplace equation)

Recall:

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right)$$

On homework 1 we showed that

$$u(x) = |x|^{\frac{p-n}{p+1}}$$

satisfies

$$\Delta_p u = 0 \quad \text{for} \quad x \neq 0.$$

Recall $-\Delta_p u$ is elliptic (Euler-Lagrange Eq for convex L)

Now suppose $u(x)$ is a smooth function satisfying

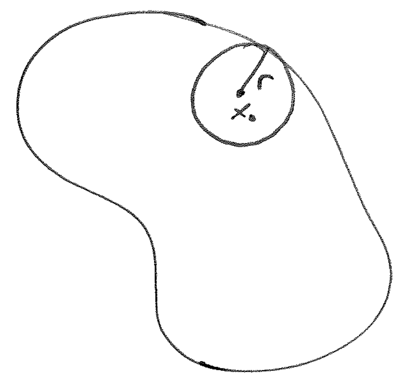
$$\begin{cases} -\Delta_p u = 0, & \text{in } U \\ u = g, & \text{on } \partial U. \end{cases}$$

when $U \subseteq \mathbb{R}^n$ is open and bounded.

Fix $x_0 \in U$ and let

$$r = \text{dist}\,(x, \partial U) = \min\{|x_0 - y| : y \in \partial U\}$$

By the comparison principle



$$\min_{x \in \partial U} g(x) \leq u(x) \leq \max_{x \in \partial U} g(x)$$

Since constant functions are solutions of $\Delta_p = 0$

Define $$V(x) = u(x_0) + C|x - x_0|^{\alpha} \qquad \alpha = \frac{p-n}{p-1}$$

and assume $p > n$. Choose C so that

$$V \geq u \quad \text{on } \partial B(x_0, r)$$

That is, we need

$$u(x_0) + C|x - x_0|^{\alpha} \geq \max_{x \in \partial U} g(x) \qquad \text{for } |x - x_0| = r$$

or

$$u(x_0) + C r^{\alpha} \geq \max_{\partial U} g$$

So set

$$C r^{\alpha} = \max_{\partial U} g - \min_{\partial U} g$$

or

$$C = r^{-\alpha}\left(\max g - \min g\right)$$

Since $v(x_0) = u(x_0)$ we can use the comparison principle on

$$B(x_0, r) \setminus \{x_0\}$$

to get that

$$u \le v \quad \text{on} \quad B(x_0, r)$$

Thus

$$u(x) \le u(x_0) + C|x - x_0|^\alpha \quad \text{for} \quad |x - x_0| \le r$$

or

$$u(x) - u(x_0) \le (\max g - \min g) \frac{|x - x_0|^\alpha}{r^\alpha}$$

for $|x - x_0| \le r$. If $|x - x_0| > r$ then

$$u(x) - u(x_0) \le (\max g - \min g)$$
$$\le (\max g - \min g) \frac{|x - x_0|^\alpha}{r^\alpha}.$$

Thus, for all $x \in U$,

$$u(x) - u(x_0) \leq (\max g - \min g) r^{-\alpha} |x - x_0|^\alpha$$

where $r = \operatorname{dist}(x_0, \partial U)$ and $\alpha = \frac{p-n}{p-1}$

We can compare against $-v$ to get

$$u(x) - u(x_0) \geq -(\max g - \min g) r^{-\alpha} |x - x_0|^\alpha$$

Hence

$$(*) \quad |u(x) - u(x_0)| \leq C |x - x_0|^\alpha \quad \text{for all } x, x_0 \in U$$

where $C = (\max g - \min g) r^{-\alpha}$

$r = \operatorname{dist}(x_0, \partial U)$

$\alpha = \frac{p-n}{p-1}$

$(*)$ says that u is Hölder continuous.

Aside: We did not prove comparison for $\Delta_p = 0$ without strict inequality $-\Delta_p u < -\Delta_p v$. This is non-trivial. One way around this is to use

$$v(x) = u(x_0) + C|x - x_0|^\alpha$$

for some $0 < \alpha < \frac{p-n}{p-1}$.

Then we can show that

$$-\Delta_p v > 0 \quad \text{for} \quad x \neq x_0.$$