Math 5588.

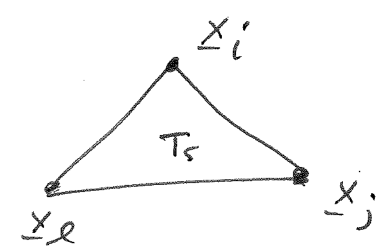
Finite Element Method cont'd...

We want to compute the functions

$$\varphi_\ell(x_i, y_i) = \begin{cases} 1, & i=\ell \\ 0, & i\neq \ell \end{cases}$$

which are extended to be piecewise affine and continuous with respect to our triangulation. Consider a triangle T_s with vertices $\underline{x}_i, \underline{x}_j, \underline{x}_\ell$ where

$$\underline{x}_i = (x_i, y_i)$$



Then on T_s

$$\varphi_\ell(x,y) = \alpha + \beta x + \gamma y \quad , \quad \begin{aligned} \alpha &= \alpha(s) \\ \beta &= \beta(s) \\ \gamma &= \gamma(s) \end{aligned}$$

To find α, β, γ

$$\varphi_\ell(\underline{x}_i) = \alpha + \beta x_i + \gamma y_i = 0$$

$$\varphi_\ell(\underline{x}_j) = \alpha + \beta x_j + \gamma y_j = 0$$

$$\varphi_\ell(\underline{x}_\ell) = \alpha + \beta x_\ell + \gamma y_\ell = 1$$

So we have to solve a linear system, which we do by row-reduction.

$$\left[\begin{array}{ccc|c} 1 & x_i & y_i & 0 \\ 1 & x_j & y_j & 0 \\ 1 & x_l & y_l & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & x_i & y_i & 0 \\ 0 & x_j - x_i & y_j - y_i & 0 \\ 0 & x_l - x_i & y_l - y_i & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & x_i & y_i & 0 \\ 0 & x_j - x_i & y_j - y_i & 0 \\ 0 & 0 & y_l - y_i - \left(\frac{x_l - x_i}{x_j - x_i}\right)(y_j - y_i) & 1 \end{array}\right]$$

Hence

$$\gamma \underbrace{\left[(y_l - y_i)(x_j - x_i) - (x_l - x_i)(y_j - y_i)\right]}_{\Delta} = x_j - x_i$$

and so $\boxed{\gamma = \dfrac{x_j - x_i}{\Delta}}$

We also have $\beta(x_j - x_i) + \dfrac{(x_j - x_i)}{\Delta}(y_j - y_i) = 0$

and so $\boxed{\beta = \dfrac{y_i - y_j}{\Delta}}$

Finally $$\alpha + \beta x_i + \gamma y_i = 0$$

as $$\alpha + \left(\frac{y_i - y_j}{\Delta}\right) x_i + \left(\frac{x_j - x_i}{\Delta}\right) y_i = 0$$

So $$\alpha = \frac{x_i y_j - x_i y_i - x_j y_i + x_i y_i}{\Delta}$$

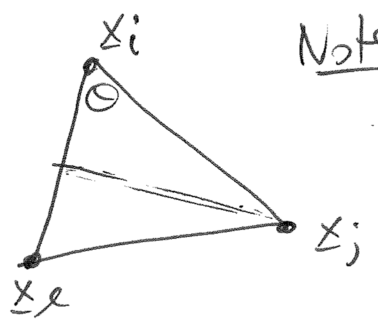
$$= \frac{x_i y_j - x_j y_i}{\Delta}$$

In summary, $$\varphi_\ell(x, y) = \alpha + \beta x + \gamma y$$

where $$\alpha = \frac{x_i y_j - x_j y_i}{\Delta}, \quad \beta = \frac{y_i - y_j}{\Delta}, \quad \gamma = \frac{x_j - x_i}{\Delta}$$

and $$\Delta = (y_\ell - y_i)(x_j - x_i) - (x_\ell - x_i)(y_j - y_i)$$

$$= (\underline{x}_j - \underline{x}_i) \times (\underline{x}_\ell - \underline{x}_i)$$



Note: $\text{Area} = \frac{1}{2} |\underline{x}_\ell - \underline{x}_i| |\underline{x}_j - \underline{x}_i| \sin\theta$

$= \frac{1}{2} |\Delta|$

On a triangle for which $\underline{x}_\ell$ is a vertex

$$\nabla \varphi_\ell(x, y) = \frac{1}{\Delta}\begin{bmatrix} \beta \\ \gamma \end{bmatrix}$$

$$= \frac{\begin{bmatrix} y_i - y_j \\ x_j - x_i \end{bmatrix}}{(\underline{x}_j - \underline{x}_i) \times (\underline{x}_\ell - \underline{x}_i)} \qquad (*)$$

On any other triangle $\nabla \varphi_\ell = 0$.

Let the triangles be $T_1, \ldots, T_M$. Then

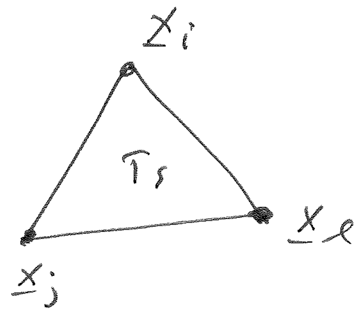
$$K_{ij} = \int_U \nabla \varphi_i \cdot \nabla \varphi_j \, dx \approx \sum_{s=1}^{M} \int_{T_s} \nabla \varphi_i \cdot \nabla \varphi_j \, dx$$

$$= \sum_{s=1}^{M} K_{ij}^{s}$$

where $K_{ij}^{s} = \int_{T_s} \nabla \varphi_i \cdot \nabla \varphi_j \, dx$

Note that $K^s_{ij} = 0$ unless $\underline{x}_i$ and $\underline{x}_j$ are both vertices of T_s.

We have 2 cases now:



① $\underline{i \neq j}$ Then on T_s

$$\nabla \varphi_i(x,y) = \frac{\begin{bmatrix} y_\ell - y_j \\ x_j - x_\ell \end{bmatrix}}{(\underline{x}_j - \underline{x}_\ell) \times (\underline{x}_i - \underline{x}_\ell)} = \frac{\begin{bmatrix} y_\ell - y_j \\ x_j - x_\ell \end{bmatrix}}{\Delta}$$

by (*) with i and ℓ swapped

Now swap i and j above to get

$$\cancel{\nabla \varphi_j(x,y) = [\, y_\ell - y_i}$$

$$\nabla \varphi_j(x,y) = \frac{\begin{bmatrix} y_\ell - y_i \\ x_i - x_\ell \end{bmatrix}}{(\underline{x}_i - \underline{x}_\ell) \times (\underline{x}_j - \underline{x}_\ell)} = \frac{\begin{bmatrix} y_i - y_\ell \\ x_\ell - x_i \end{bmatrix}}{\Delta}$$

since $\quad V_1 \times V_2 = -V_2 \times V_1$

[Note: Actually $V_1 \times V_2$ is a vector in $(0,0,1)$ direction. We are identifying $V_1 \times V_2$ with the signed magnitude]

Therefore

$$\nabla\varphi_i \cdot \nabla\varphi_j = \frac{(y_\ell - y_j)(y_i - y_\ell) + (x_j - x_\ell)(x_\ell - x_i)}{\Delta^2}$$

$$= -\frac{(\underline{x}_i - \underline{x}_\ell)\cdot(\underline{x}_j - \underline{x}_\ell)}{4A_s^2}$$

where $A_s = \text{Area}(T_s) = \frac{1}{2}|\Delta|$

Finally,

$$K_{ij}^s = \int_{T_s} \nabla\varphi_i \cdot \nabla\varphi_j \, dx = -\frac{(\underline{x}_i - \underline{x}_\ell)\cdot(\underline{x}_j - \underline{x}_\ell)}{4A_s^2} \cdot A_s$$

or

$$\boxed{K_{ij}^s = -\frac{(\underline{x}_i - \underline{x}_\ell)\cdot(\underline{x}_j - \underline{x}_\ell)}{4A_s}}$$

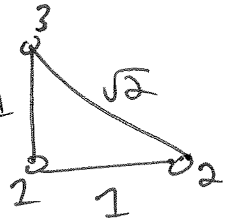
If $i=j$ then

$$\boxed{K_{ii}^s = \frac{(y_j - y_\ell)^2 + (x_j - x_\ell)^2}{\Delta^2} A_s = \frac{|\underline{x}_j - \underline{x}_\ell|^2}{4A_s}}$$

Elemental stiffness.

Each triangle associated with 6 values K_{ij} $i \le j$ $(K_{ij} = K_{ji})$

These are called elemental stiffness

Example (1) 

Area $= \frac{1}{2}$

$$K_{11} = \frac{(\sqrt{2})^2}{4A} = \frac{2}{2} = 1$$

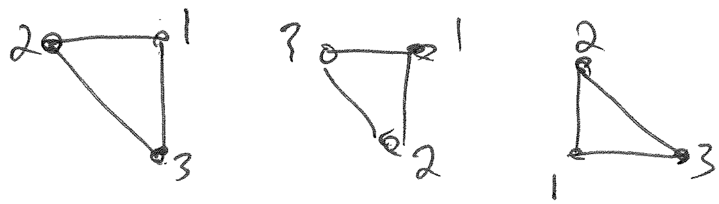
These are K_{ij}s

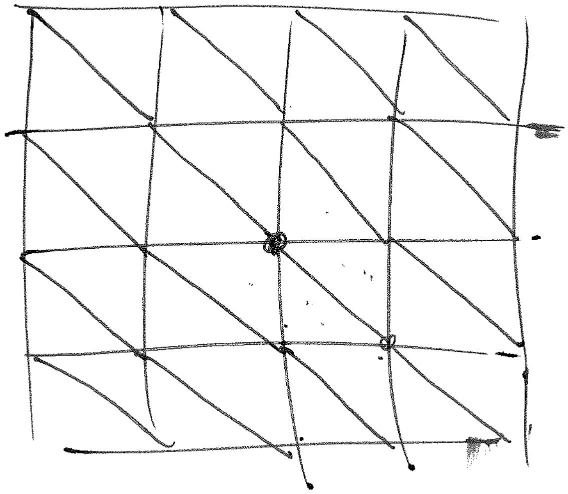
$$K_{22} = \frac{1}{4A} = \frac{1}{2} = K_{33}$$

$$K_{12} = -\frac{(0,-1)\cdot(1,-1)}{4A} = -\frac{1}{2} = K_{21}$$

$$K_{13} = -\frac{(1,0)\cdot(1,-1)}{4A} = -\frac{1}{2} = K_{31}$$

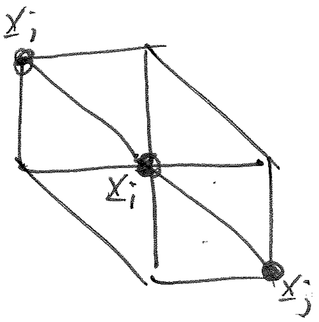
$$K_{23} = -\frac{(0,1)\cdot(1,0)}{4A} = 0 = K_{32}$$

Same for 



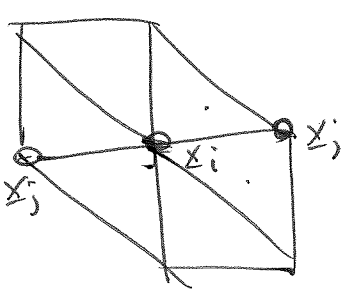
$K_{ij} = 0$ unless $i=j$ or i,j vertices of same triangle.

(1)



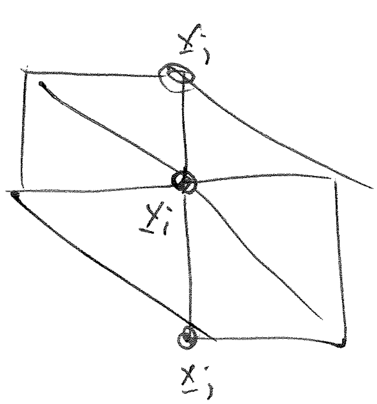
$K_{ij} = 0$

(2)



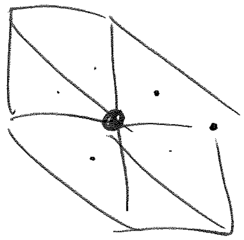
$K_{ij} = -1$

(3)



$K_{ij} = -1$

(4)



$$K_{ii} = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$$

(5) So, the equation to be solved is

$f = 1$

$$4U_{ij} - U_{ij-1} - U_{i-1j} - U_{i+1j} - U_{ij+1} = \cancel{6(\tfrac{1}{2}h^2)}$$

$$\approx \int_{P_{ij}} f(x)$$

$$= \int_{\mathbb{R}^2} \varphi_{ij} f \, dx = h^2 f_{ij} + O(h^3)$$

So reduces to finite difference in this case.