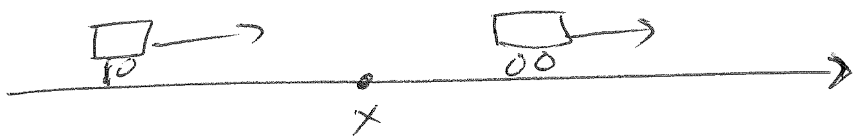
Math 5588 4/6/17 and 4/11/17

Scalar Conservation Laws

Example (Traffic flow)



Let $u(x,t)$ = density of cars (cars/mile) on a highway at (x,t)

Let $V(x,t)$ = velocity (miles/hour) at (x,t)

$F(x,t)$ = traffic flow (cars/hour) passing by position x at time t.

Note:

$$F(x,t) = u(x,t) \cdot V(x,t) \qquad (*)$$

$$\left(\frac{\text{cars}}{\text{hour}}\right) \qquad \left(\frac{\text{cars}}{\text{mile}}\right) \times \left(\frac{\text{miles}}{\text{hour}}\right)$$

and

$$\begin{array}{l}\#\text{ cars between} \\ x=a \text{ and } x=b\end{array} = \int_a^b u(x,t)\,dx$$

Assuming traffic is a conserved quantity (there are no on/off ramps so cars are neither created nor destroyed) we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = F(a,t) - F(b,t)$$
$$= (\text{Flow in} - \text{Flow out})$$

Set $a=x$ and $b=x+h$ for $h>0$ small. Then

$$\frac{1}{h}\int_x^{x+h} u_t(s,t)\,ds = \frac{F(x,t) - F(x+h,t)}{h}$$

Send $h \longrightarrow 0^+$ to get

$$u_t(x,t) = -F_x(x,t)$$

Hence

$$\boxed{u_t(x,t) + F_x(x,t) = 0}$$

This holds for any conserved quantity (heat, gas, fluids) and is called the <u>mass continuity equation</u>.

Recalling (*) we have

$$u_t + \frac{\partial}{\partial x}(uV) = 0$$

for traffic flow. It is common to assume the velocity $V(x,t)$ is a function of the density $u(x,t)$:

$$V(x,t) = v(u(x,t))$$

Idea is that when $u \approx 0$ road is clear and V is large, but as u increases, velocity decreases as traffic begins to slow and eventually a traffic jam forms. Generally V looks like

Hence

$$\boxed{u_t + \frac{\partial}{\partial x}(u\,v(u)) = 0}$$

or

$$u_t + u_x v(u) + v'(u) u u_x = 0$$

We take $v(u) = 1 - u$ in simulations.

Setting $\rho(x,t) = u(x,t)$ we have

$$\rho_t + [\rho(1-\rho)]_x = 0 \qquad \rho = \text{density}$$

Make the change of variables

$$u(x,t) = 1 - 2\rho(x,t)$$

Then $\quad \rho = \frac{1}{2} - \frac{u}{2}$

$$1 - \rho = \frac{1}{2} + \frac{u}{2}$$

$$[\rho(1-\rho)]_x = \left(\frac{1}{4} - \frac{u^2}{4}\right)_x = -\frac{u}{2} u_x$$

$$\rho_t = -\frac{u_t}{2}$$

Hence

$$-\frac{u_t}{2} - \frac{u u_x}{2} = 0$$

or

$$\boxed{u_t + u u_x = 0}$$

This is the inviscid Burger's equation.

It is actually a simple one dimensional version of the inviscid Navier-Stokes (or Euler) equations.

Note Burger's equation is

$$(u_x, u_t) \cdot (u, 1) = 0$$

So solutions are constant along lines with speed

$$\frac{dx}{dt} = \frac{u}{1} = u.$$

These are the characteristics for Burger's equation. The characteristics depend on the solution u!

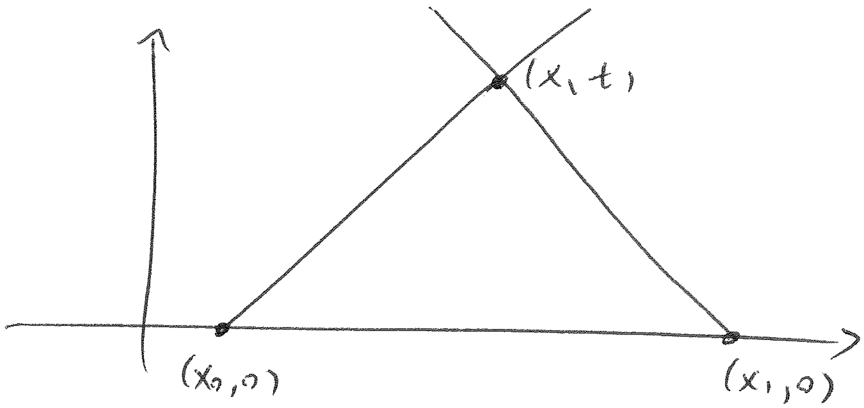
Note if
$$\frac{dx}{dt} = u(x(t), t)$$

then
$$\frac{d}{dt} u(x(t), t) = u_x \frac{dx}{dt} + u_t$$
$$= u u_x + u_t = 0$$

Hence u is constant along characteristics, and so the characteristics are straight lines.

Solving Burger's equation via method of characteristics.

$$\begin{cases} u_t + u u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) \end{cases}$$



$$\frac{x - x_0}{t - \cancel{t_0}} = \frac{dx}{dt} = u(x, t) = u(x_0, 0) = f(x_0)$$

Hence $$\boxed{x - x_0 = t f(x_0)} \quad (*)$$

If we can solve $x_0 = x_0(x, t)$

then $$u(x, t) = f(x_0(x, t))$$

<u>Example</u>: $f(x) = x^2$. Then $(*)$

becomes

$$x - x_0 = t x_0^2$$

or $$t x_0^2 + x_0 - x = 0$$

$$X_0 = \frac{-1 \pm \sqrt{1+4tx}}{2t}$$

$$= -\frac{1}{2t} \pm \frac{1}{2t}\sqrt{1+4xt} \qquad t \neq 0$$

Hence

$$u(x,t) = f(x_0(x,t))$$

$$= \left(\frac{-1 \pm \sqrt{1+4xt}}{2t} \right)^2$$

$$= \frac{1 \mp 2\sqrt{1+4xt} + 1 + 4xt}{4t^2}$$

$$= \frac{1 + 2xt \mp \sqrt{1+4xt}}{2t^2} \qquad \text{for } t \neq 0$$

Note not defined at $t=0$. We require

$$\lim_{t \to 0} u(x,t) = x^2$$

Hence need to take negative root:

$$\lim_{t \to 0^+} \frac{1 + 2xt - \sqrt{1 + 4xt}}{2t^2}$$

$$= \lim_{t \to 0^+} \frac{2x - 2x(1+4xt)^{-1/2}}{4t} \qquad \text{(l'Hospital's)}$$

$$= \lim_{t \to 0^+} \frac{4x^2(1+4xt)^{-3/2}}{4} = x^2.$$

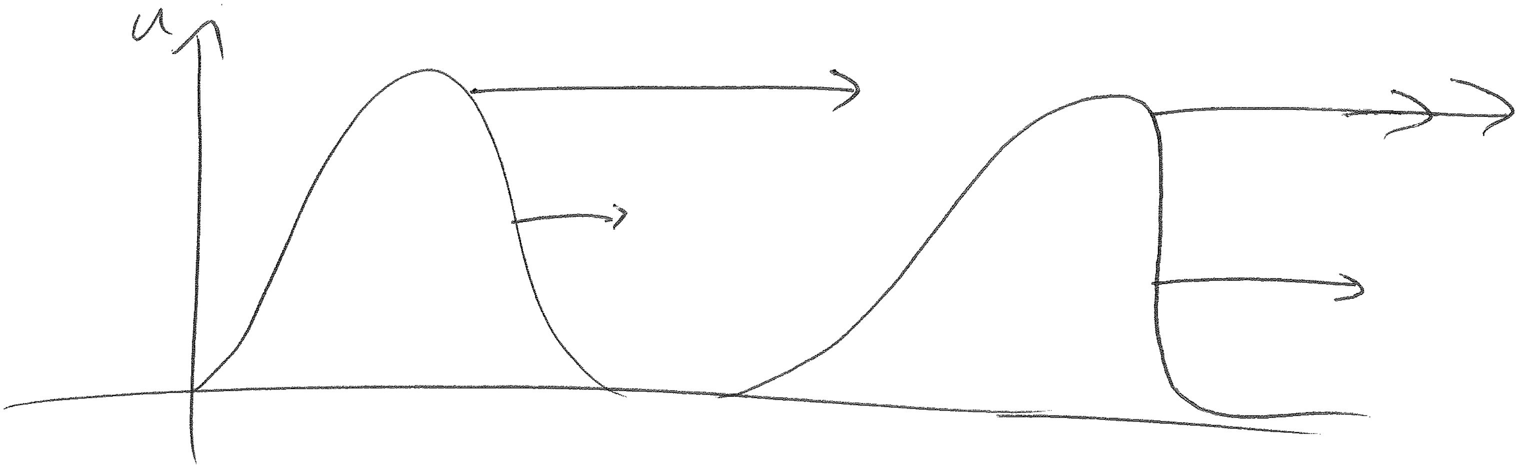
Hence the solution is

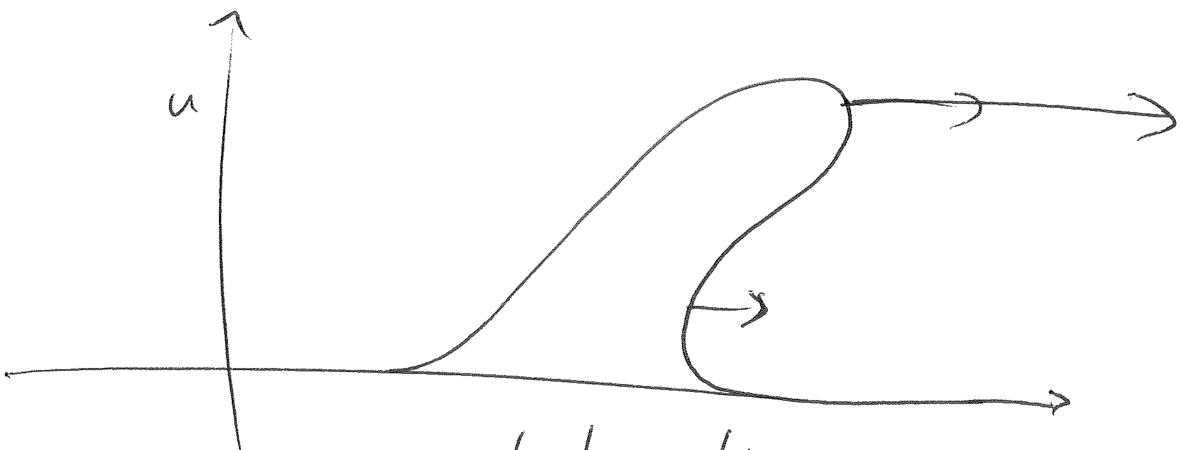
$$\begin{cases} u(x,t) = \dfrac{1 + 2xt - \sqrt{1+4xt}}{2t^2}, & t \neq 0 \\ u(x,0) = x^2 \end{cases}$$

Recall Burger's eq.

$$u_t + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

Velocity is $V = u$.





later time.

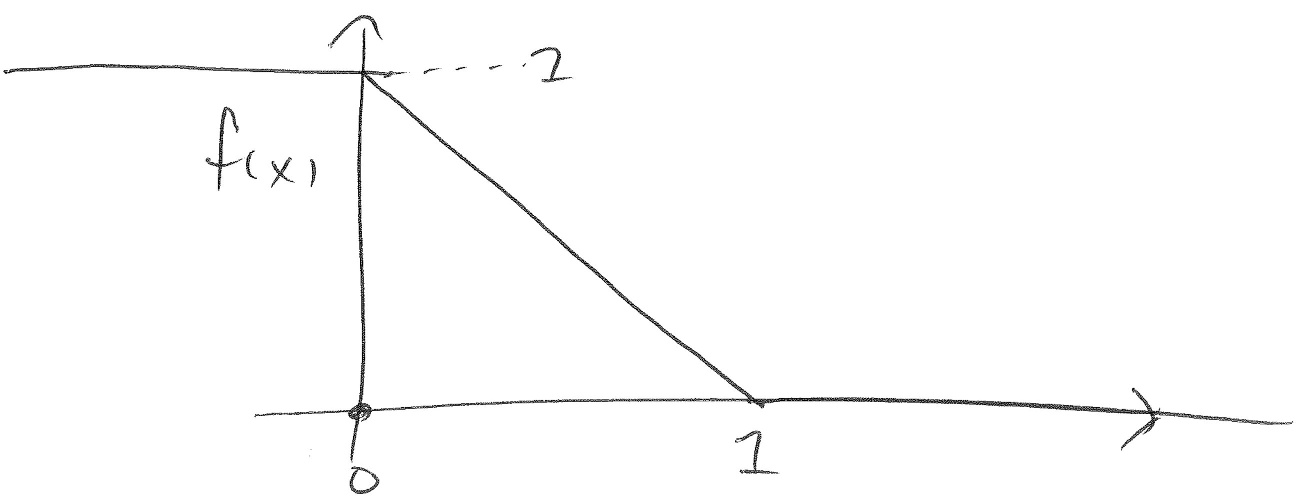
Solution can become <u>multivalued</u> when characteristics cross.

How to deal with this.

Example (Burger's equation)

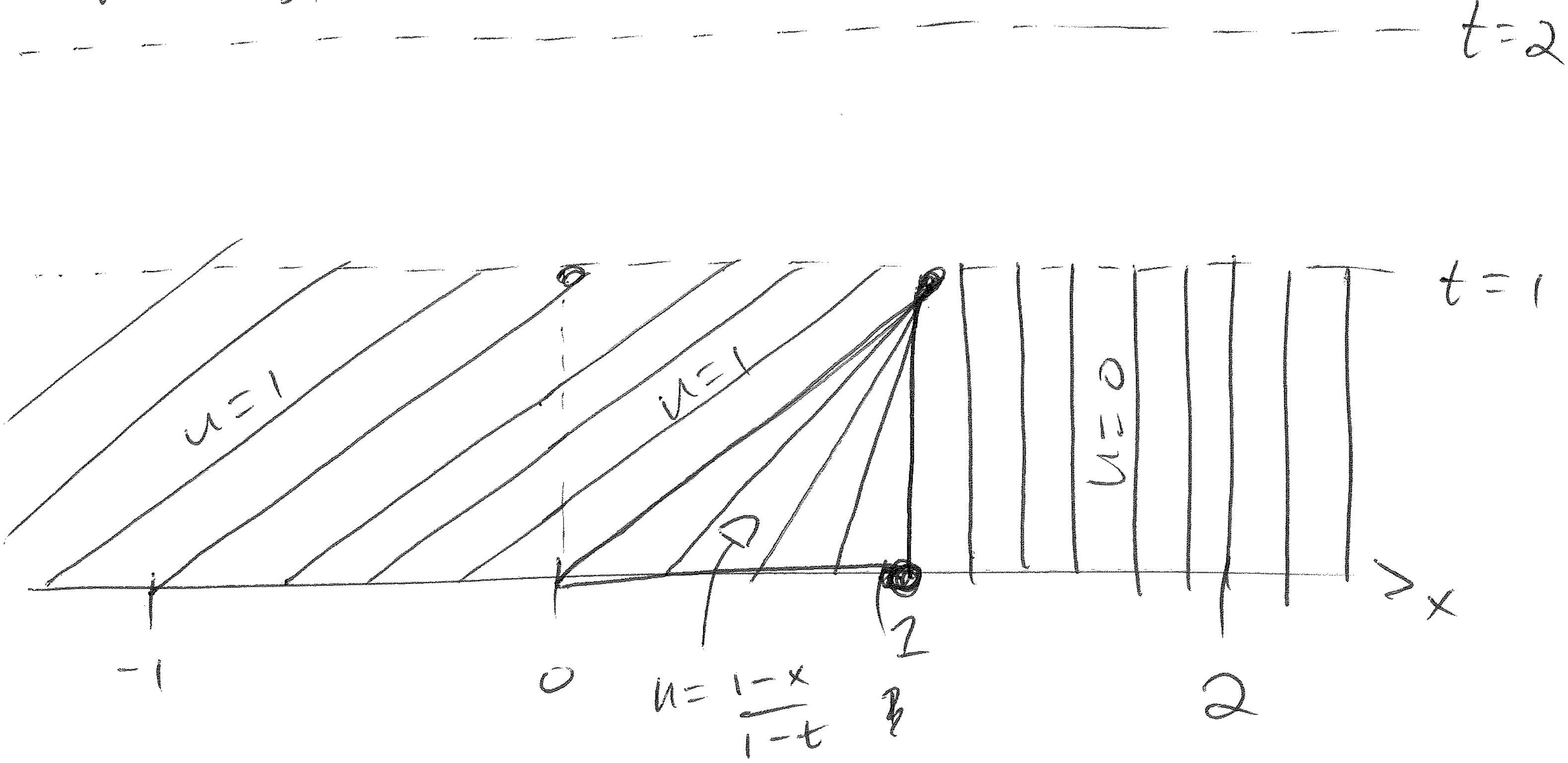
$$\begin{cases} u_t + u u_x = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & x \geq 1 \end{cases}$$



This is supposed to mimic illustration from ~~last~~ previous page.

We sketch the characteristics now.



Between $0 \le x \le 1$, speed $\frac{dx}{dt} = 1 - x$

So at $t = 1$, $\frac{x - x_0}{1 - 0} = 1 - x_0$

reduces to $x = 1$. So all characteristics between $0 \le x \le 1$ meet at the same point $x = 1$, $t = 1$. In the wedge

$0 \le x \le 1$ and $t \le x$

we have

$$\frac{x - x_0}{t - 0} = \frac{dx}{dt} = u(x, t) = 1 - x_0$$

$$x - x_0 = t(1 - x_0)$$

$$x = t + x_0(1 - t)$$

$$x_0 = \frac{x - t}{1 - t}$$

Hence $u(x,t) = 1 - x_0 = \dfrac{1 - t - (x - t)}{1 - t}$

$$= \frac{1 - x}{1 - t}$$

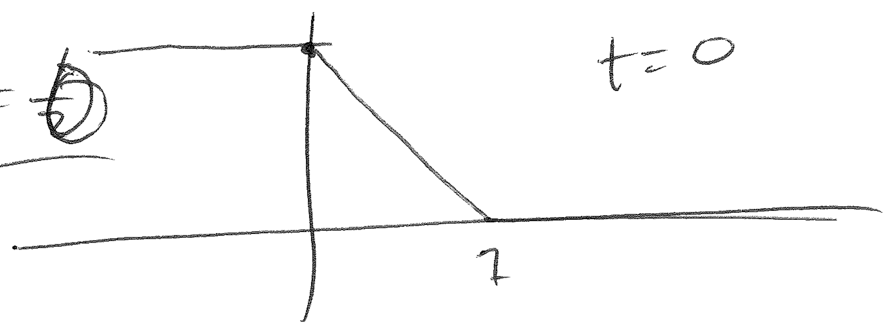
Hence for $0 \le t \le 1$

$$u(x,t) = \begin{cases} 1, & x \le t \le 1 \\ \dfrac{1 - x}{1 - t}, & 0 \le t \le x \\ 0, & x \ge 1 \end{cases}$$

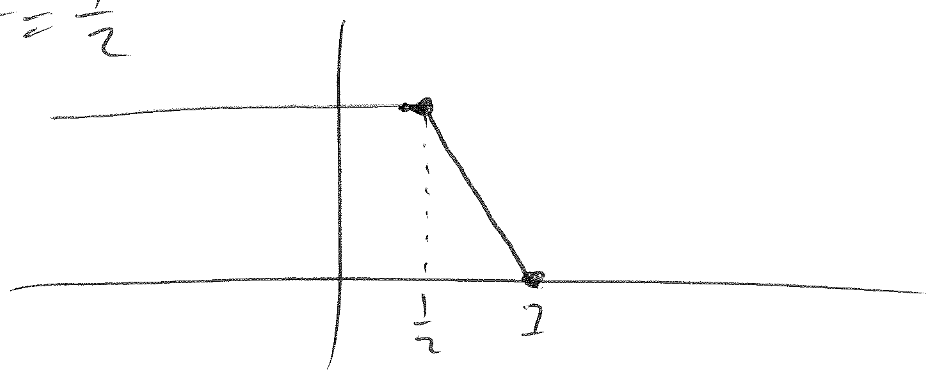
__Question__: What does the solution look like for $t \geq 1$?

__Answer__: A shock-wave forms, and we must understand the solution in a weaker sense.

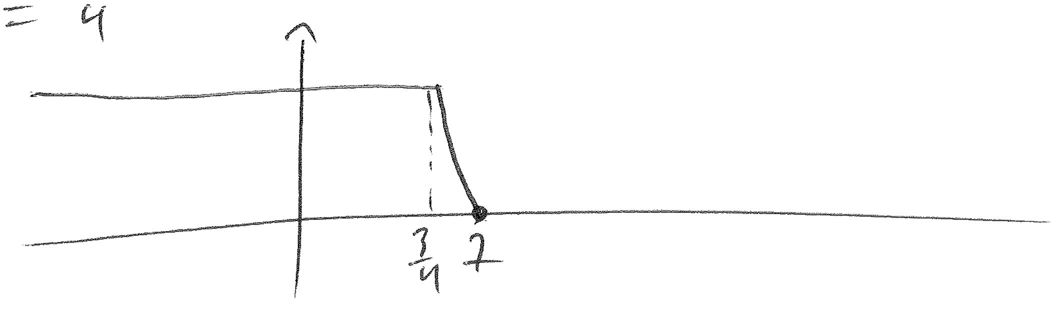
__Note__: $t = 0$

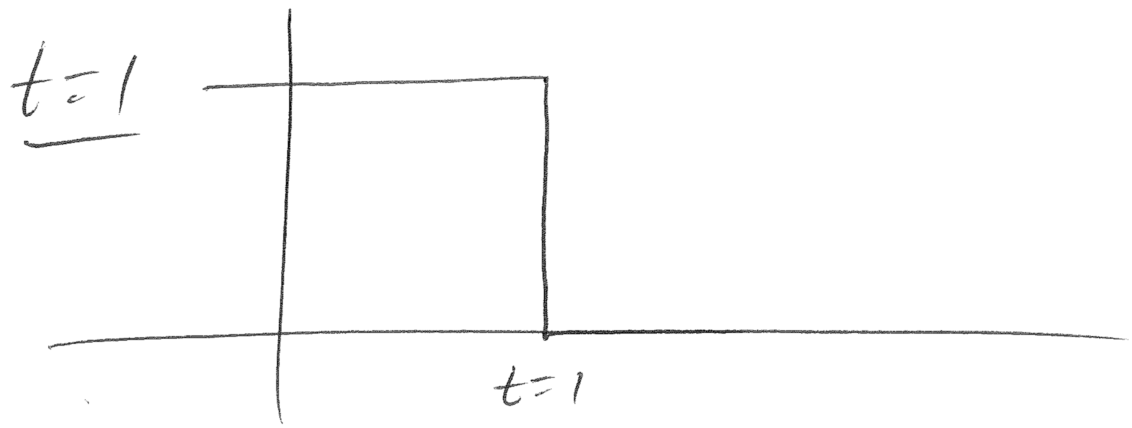


$t = \frac{1}{2}$



$t = \frac{3}{4}$





__Weak solutions of scalar conservation laws__

Consider the general conservation law

$$(*) \begin{cases} u_t + [F(u)]_x = 0, & t > 0 \\ u(x,0) = f(x) \end{cases}$$

Let $\varphi(x,t)$ be smooth with compact support (vanishing outside $|x|^2 + t^2 \leq R^2$ for some R). Then

$$u_t \varphi + [F(u)]_x \varphi = 0$$

So

$$0 = \int_0^\infty \int_{-\infty}^\infty u_t \varphi + [F(u)]_x \varphi \, dx\, dt$$

$$= \int_{-\infty}^\infty \left(\int_0^\infty u_t \varphi \, dt \right) dx + \int_0^\infty \left(\int_{-\infty}^\infty [F(u)]_x \varphi \, dx \right) dt$$

$$= \int_{-\infty}^\infty \left[u\varphi \Big|_{t=0}^\infty - \int_0^\infty u \varphi_t \, dt \right] dx$$

$$+ \int_0^\infty \left[\underbrace{F(u)\varphi \Big|_{x=-\infty}^\infty}_{0} - \int_{-\infty}^\infty F(u) \varphi_x \, dx \right] dt$$

$$= \int_{-\infty}^\infty -f(x)\varphi(x)\, dx - \int_0^\infty \int_{-\infty}^\infty u \varphi_t + F(u) \varphi_x \, dx\, dt$$

Since φ has compact support and

$$u(x,0) = f(x)$$

Def: We say $u(x,t)$ is a *weak solution* of (*) if for all smooth compactly supported $\varphi(x,t)$

$$\boxed{\int_0^\infty \int_{-\infty}^{\infty} u\varphi_t + F(u)\varphi_x \, dx\, dt + \int_{-\infty}^{\infty} f \varphi \, dx = 0}$$

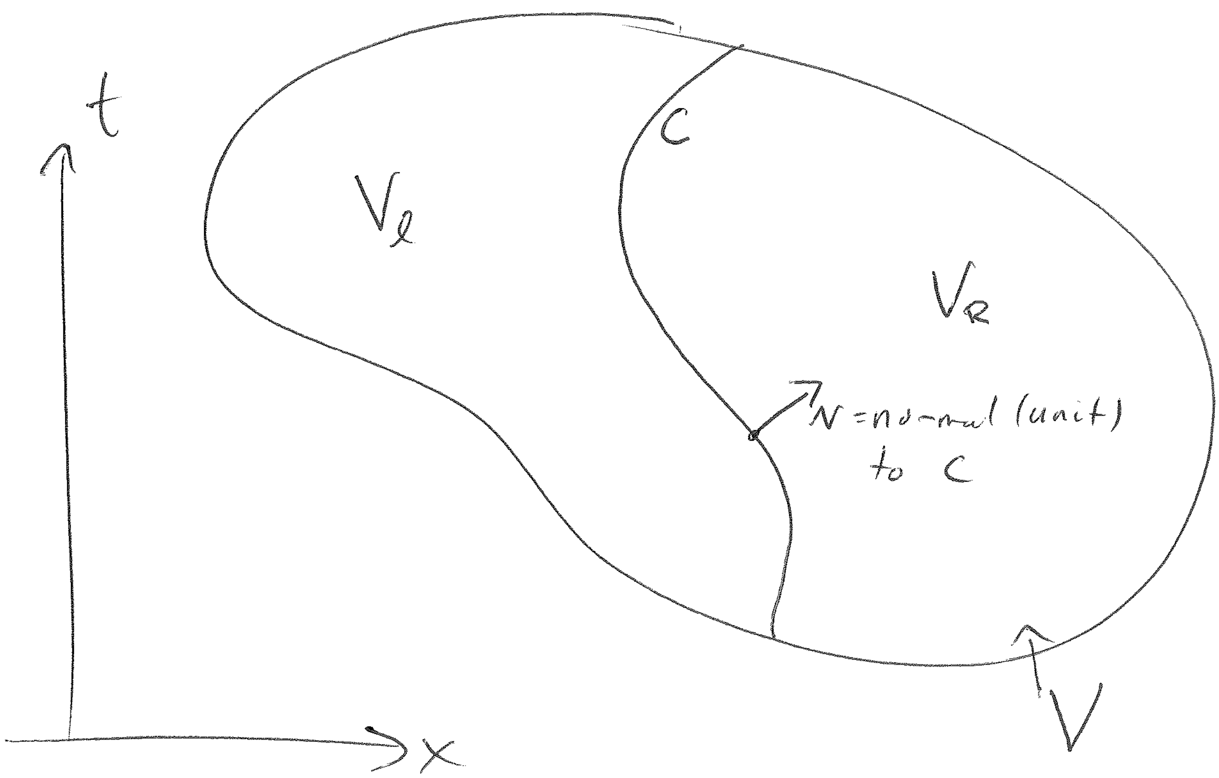
The notion of weak solution allows discontinuous functions to be solutions of conservation laws.

Any continuous and piecewise C^1 function, with C^1 regions separated by finite collection of smooth curves, that solves the PDE in the usual sense in each C^1 region is a weak solution (exercise: prove this).

What properties of solutions can we deduce from the weak form?

Suppose we have a piecewise C^1 solution with a discontinuity along a curve

$$C = (x(t), t)$$



So $\quad u_t + F(u)_x = 0 \quad$ in $\quad V_R$

$\quad u_t + F(u)_x = 0 \quad$ in $\quad V_l$

and $\quad u$ is a weak solution in V.

Choose a smooth test function $\varphi(x,t)$ with compact support in V (so $\varphi(x,t) = 0$ on ∂V).

Note φ need <u>not</u> vanish on $\partial V_\ell \cap \partial V_R$.

Then

$$0 = \int_0^\infty \int_{-\infty}^\infty u\varphi_t + F(u)\varphi_x \, dx\, dt$$

$$= \iint_{V_\ell} u\varphi_t + F(u)\varphi_x \, dx\, dt + \iint_{V_R} u\varphi_t + F(u)\varphi_x \, dx\, dt$$

$$=: A + B$$

Integrating by parts

$$A = \iint_{V_\ell} u\varphi_t + F(u)\varphi_x \, dx\, dt$$

$$= -\iint_{V_\ell} \varphi(u_t + F(u)_x)\, dx\, dt$$

$$+ \int_C \varphi(u_\ell \nu_2 + F(u_\ell)\nu_1^{\ell})\, ds$$

Recall Gauss-Green theorem

$$\int_U u_{x_i}\, dx = \int_{\partial U} u \nu_i \, dS$$

$$\int_U u v_{x_i}\, dx = \int_{\partial U} u v \nu_i \, dS - \int_U u_{x_i} v \, dx$$

Hence $$A = \int_C \varphi\left(u_\ell \nu_2 + F(u_\ell)\nu_1\right) dS$$

where $u_\ell = u(x,t)$ on left side of C

$u_R = u(x,t)$ on right side of C

[i.e., limit from left / right respectively].

Likewise,

$$B = -\iint_{V_R} \varphi\left(u_t + F(u)_x\right) dx\, dt$$

$$+ \int_C \varphi\left(u_R(-\nu_2) + F(u_R)(-\nu_1)\right) dS$$

Thus $B = -\int_C \varphi \left(u_R \nu_2 + F(u_R)\nu_1 \right) ds$

Since u is a weak solution we have

$$A + B = 0$$

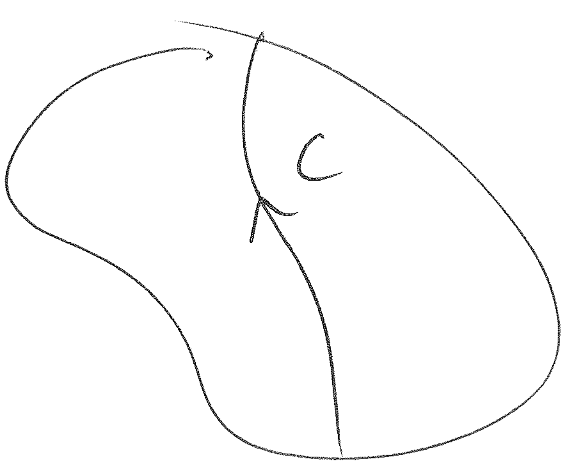
or

$$0 = \int_C \varphi \Big((u_\ell - u_R)\nu_2 + \big(F(u_\ell) - F(u_R)\big)\nu_1 \Big) ds$$

By vanishing lemma

$$(u_\ell - u_R)\nu_2 + \big(F(u_\ell) - F(u_R)\big)\nu_1 = 0$$

Note $C = (x(t), t)$ so $\nu = \dfrac{(1, -x'(t))}{\sqrt{1 + x'(t)^2}}$



Hence

$$F(u_\ell) - F(u_R) = x'(t)(u_\ell - u_R)$$

Therefore, the speed of the characteristic curve $\sigma := X'(t)$ must be

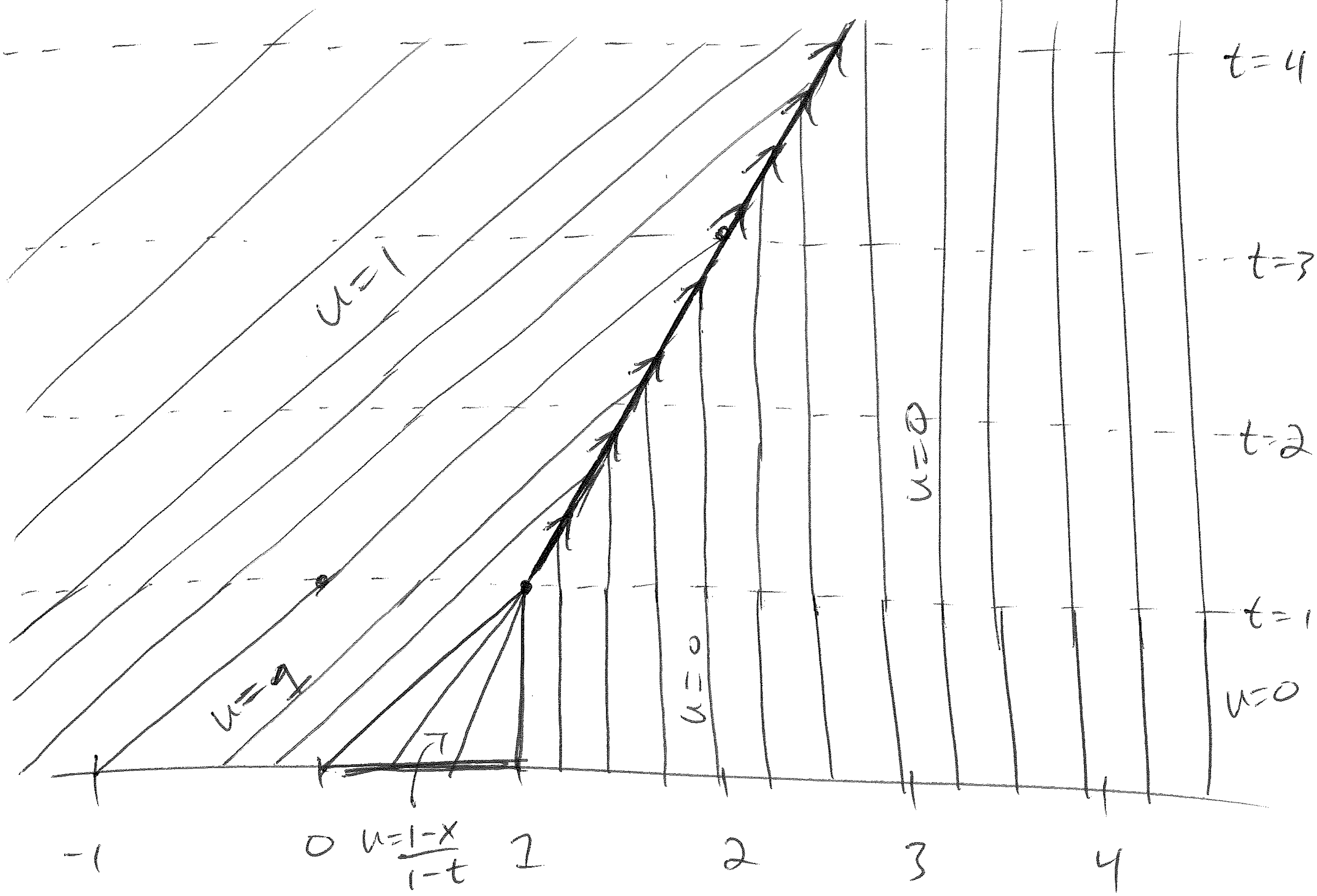
$$\sigma = \frac{F(u_\ell) - F(u_R)}{u_\ell - u_R} \quad \left(\begin{array}{c}\text{Rankine} \\ \text{Hugoniot} \\ \text{Condition}\end{array}\right)$$

$$= \frac{\text{Change in flow}}{\text{change in density}}$$

Back to example $\begin{cases} u_t + u u_x = 0, & t > 0 \\ u(x,0) = f(x) \end{cases}$

$$F(u) = \frac{1}{2}u^2, \quad f(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & x \ge 1 \end{cases}$$

$$u(x,t) = \begin{cases} 1, & x \le t \le 1 \\ \frac{1-x}{1-t}, & 0 \le t \le x \\ 0, & x \ge 1 \end{cases}$$



After $t=1$ a shockwave forms with

$u_\ell = 1 \quad , \quad u_R = 0 \quad$ at $(x,t)=(1,1)$

Shock wave moves with speed

$$\sigma = \frac{F(u_\ell) - F(u_R)}{u_\ell - u_R}$$

$$= \frac{\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2}{1-0} = \frac{1}{2}$$

Shock wave is the curve

$$X(t) = \frac{1}{2}t + \frac{1}{2}$$

Hence for $t \geq 1$

$$u(x,t) = \begin{cases} 1, & x \leq \frac{t+1}{2} \\ 0, & x > \frac{t+1}{2} \end{cases}$$

The discontinuous function u is a <u>weak</u> solution of Burger's equation.

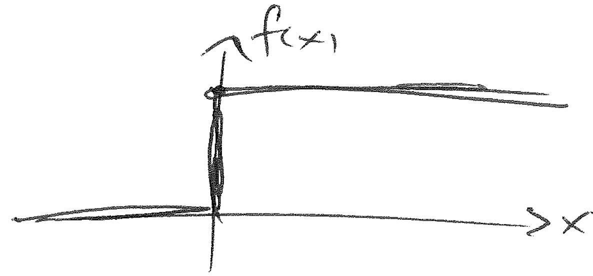
So is the notion of weak solution sufficient to uniquely describe solutions past singularities?

<u>Answer</u> No.

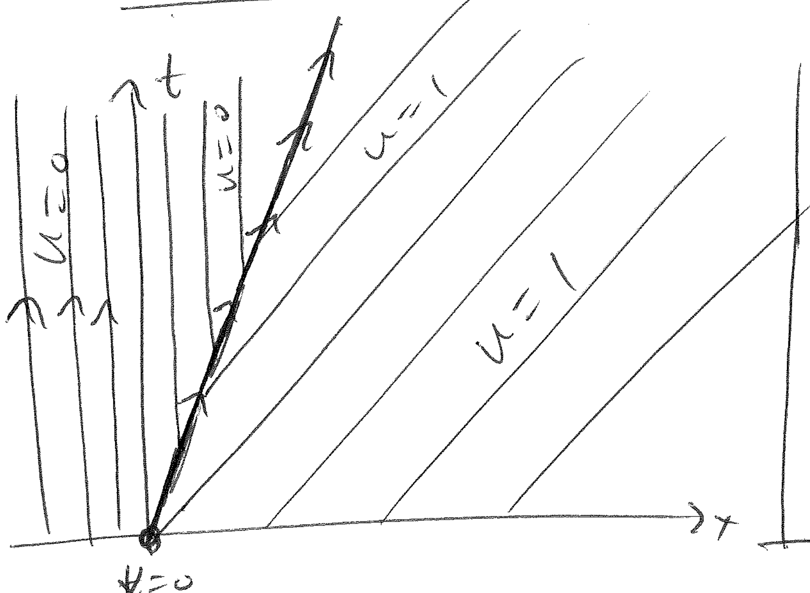
Weak Solutions are not unique!

Example: $\begin{cases} u_t + uu_x = 0 & t > 0 \\ u(x,0) = f(x) \end{cases}$

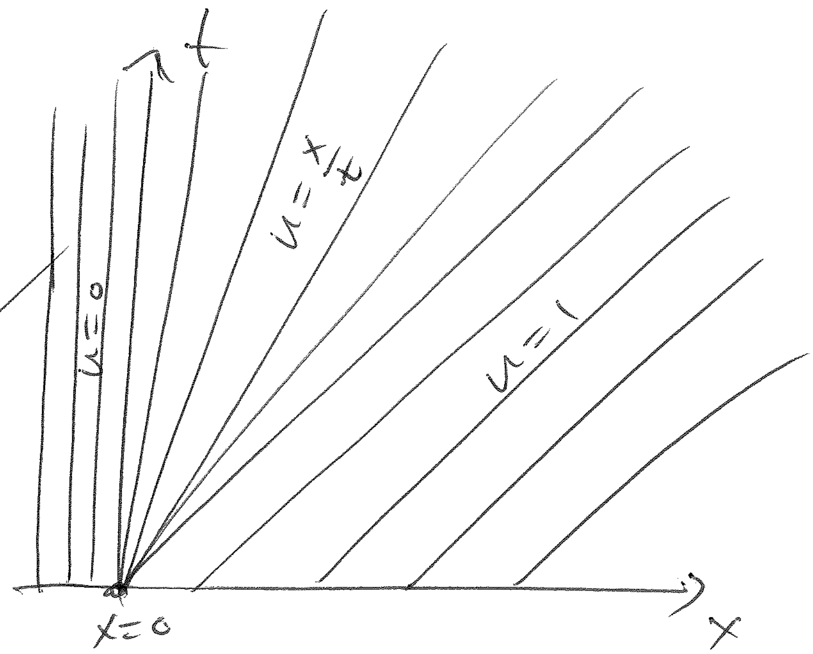
$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$



Two weak solutions



Shock wave



Rarefaction wave

$$\text{Speed} = \sigma = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r}$$

$$= \frac{\frac{1}{2}(0)^2 - \frac{1}{2}(1)^2}{0 - 1}$$

$$= \frac{1}{2}$$

$$\frac{dx}{dt} = \frac{x}{t} = u(x,t)$$

in wedge $\begin{cases} 0 \le x \le 1 \\ t \ge x \end{cases}$

Which solution (Rarefaction or shock) is physically correct?

Recall traffic flow

$$u = 1 - 2\rho$$

or

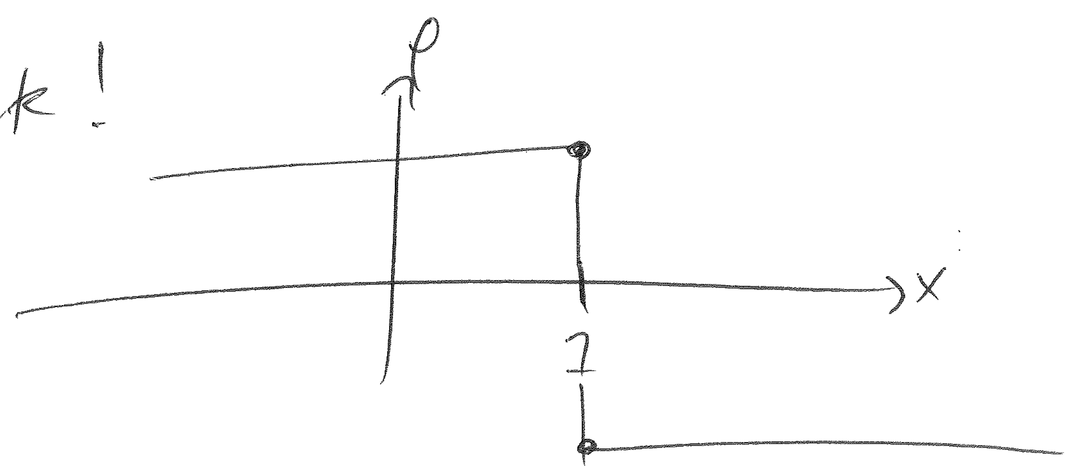
$$\rho = \frac{1}{2} - \frac{1}{2}u$$

For the shock wave

$$\rho(x,t) = \begin{cases} \frac{1}{2}, & x \le t \\ -\frac{1}{2}, & x \ge t \end{cases}$$

So traffic more dense behind the shock!

$t=1$



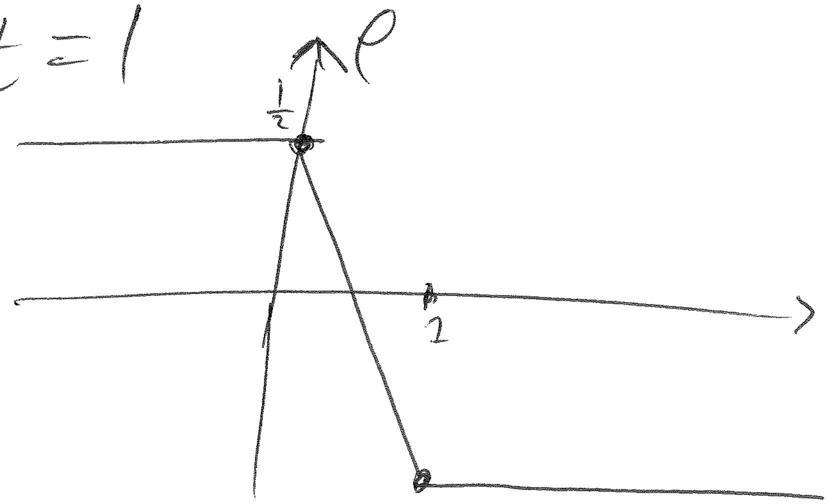
Such a shock would <u>not</u> persist in the real physical world, and would instead dissipate as time moved forward, exactly like what the rarefaction wave does.

$$\rho(x,t) = \frac{1}{2} - \frac{1}{2}\frac{x}{t}$$

$$= \frac{t-x}{2t}$$

for rarefaction wave in wedge $0 \le x \le 1$, $t \ge x$

$t = 1$



So the shock wave is a <u>non-physical</u> weak solution in this case.

For traffic flow, a <u>shock wave</u> is physical if

$$\rho_\ell \le \rho_R$$

that is, the density of traffic is greater in front of the shock, since this is exactly what caused the shock!

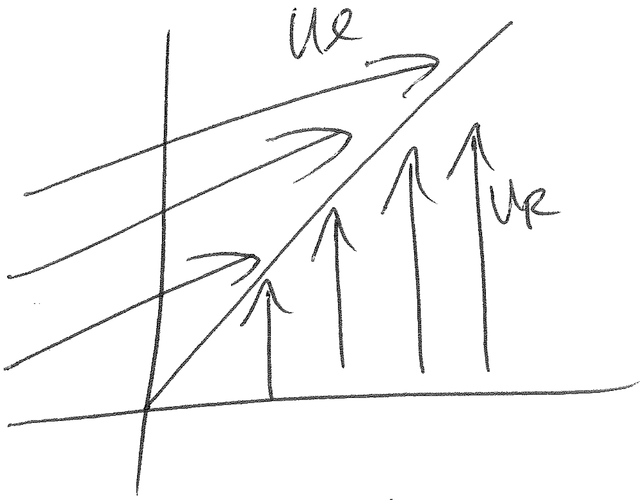
Since $u = 1 - 2\rho$

$$u_\ell = 1 - 2\rho_\ell \ge 1 - 2\rho_R = u_R$$

Hence for Burger's eq. we ask that

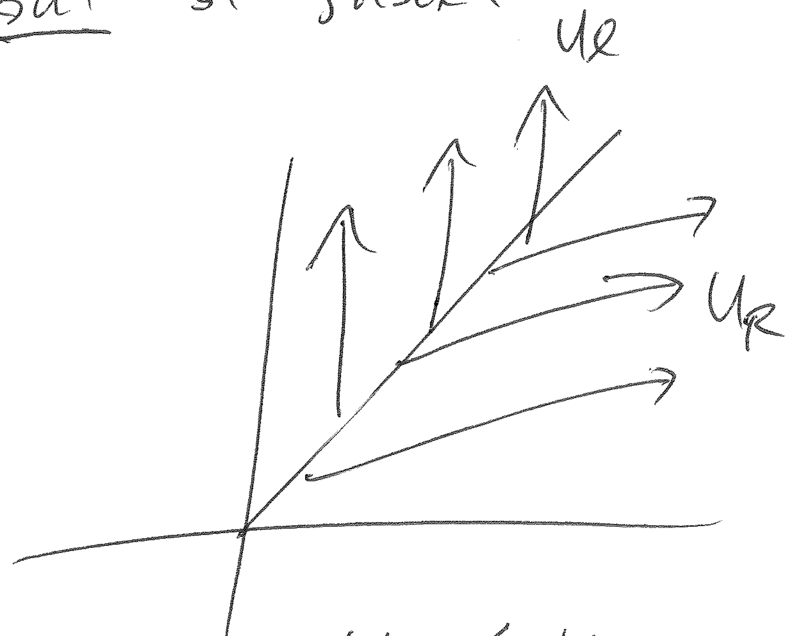
$$\boxed{u_\ell \ge u_R \quad \text{along shock curve}}$$

This means characteristics flow <u>into</u> shocks and not <u>out</u> of shocks.



$u_\ell \geq u_R$

Physical shock



$u_\ell \leq u_R$

Non-physical shock.

We call a weak solution of Burger's equation an <u>entropy</u> solution if

$$\boxed{u_\ell \geq u_R \text{ along all shocks}}$$

Entropy solutions are unique.

For a general conservation law

$$u_t + F(u)_x = 0$$

we have

$$u_t + F'(u)\, u_x = 0$$

Characteristics are given by

$$\frac{dx}{dt} = F'(u) = \text{speed}$$

Hence in general, <u>entropy solutions</u> must satisfy

$$F'(u_L) \geq F'(u_R)$$

If F is <u>strongly convex</u> $\left(F'' \geq \theta > 0\right)$ then this is equivalent to $\boxed{u_L \geq u_R}$ as F' is <u>strictly increasing</u>.