Another Example

Solve Burger's equation

$$u_t + u u_x = 0$$

with initial condition

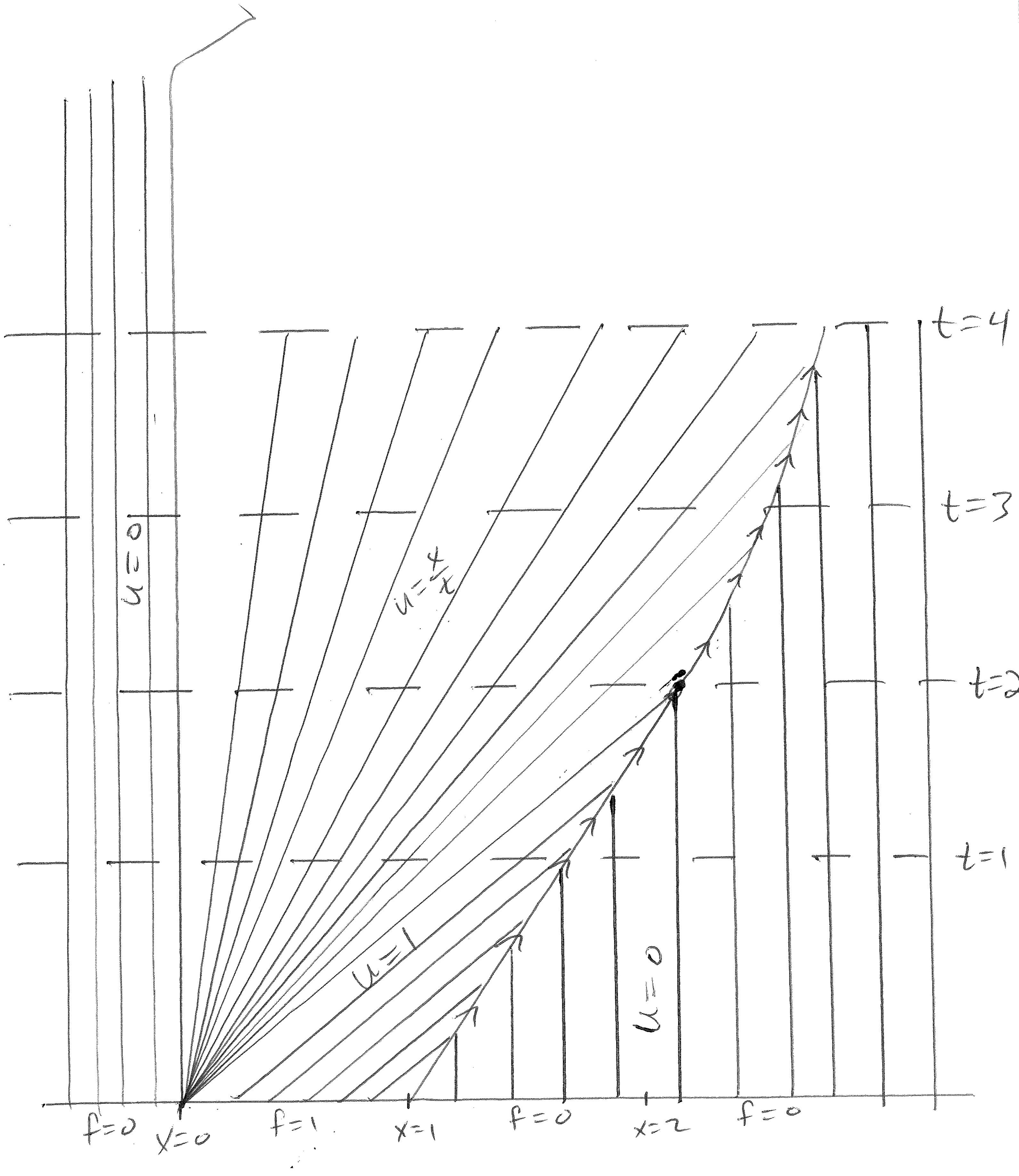
$$u(x,0) = f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Shock wave will form immediately starting at $x = 1$ and $t = 0$ $(u_L > u_R)$

Rarefaction wave $u = \frac{x}{t}$ at $x = 0, t = 0$

Shock wave has speed $F(u) = \frac{1}{2}u^2$

$$\frac{dx}{dt} = \frac{F(u_L) - F(u_R)}{u_L - u_R} = \frac{\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2}{1 - 0} = \frac{1}{2}$$


t=4
t=3
t=2
t=1
u=0
u=x/t
u=1
u=0
f=0
x=0
f=1
x=1
f=0
x=2
f=0

After $t=2$ shock wave has speed

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\frac{1}{2}\left(\frac{x}{t}\right)^2 - \frac{1}{2}(0)}{\frac{x}{t} - 0} = \dfrac{x}{2t} \\ x(2) = 2 \end{cases}$$

Solving for $x(t)$ yields $\boxed{x(t) = (2t)^{1/2}}$

Solution is for $\underline{0 \leq t \leq 2}$

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x \leq t \\ 1, & t \leq x \leq 1 + \frac{t}{2} \\ 0, & x \geq 1 + \frac{t}{2} \end{cases}$$

For $t \geq 2$

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x \leq (2t)^{1/2} \\ 0, & x \geq (2t)^{1/2} \end{cases}$$