Math 5588 4/18/2017

Hamilton-Jacobi Equations

Recall we are studying the Hamilton-Jacobi equation

$$(H) \begin{cases} u_t + H(u_x) = 0, & -\infty < x < \infty \\ & t > 0 \\ u(x,0) = f(x) \end{cases}$$

where we assume $H(p)$ is

(Convex) $H''(p) > 0$

(Superlinear) $\lim\limits_{|p| \to \infty} \dfrac{H(p)}{|p|} = +\infty$

Last time we derived the Hopf-Lax formula

$$(H\text{-}L) \quad u(x,t) = \min_{y \in \mathbb{R}} \left\{ f(y) + t L\left(\frac{x-y}{t}\right) \right\}, \quad L = H^*$$

for solutions of (H). This time we will prove that the Hopf-Lax formula (H-L) actually solves (H) in the viscosity sense.

<u>Recall</u>: H^* is the Legendre transform of H defined by

$$H^*(q) = \max_{p \in \mathbb{R}} \{ pq - H(p) \}$$

If H is <u>convex</u> and <u>superlinear</u> then

$$\boxed{H^{**} = (H^*)^* = H}$$

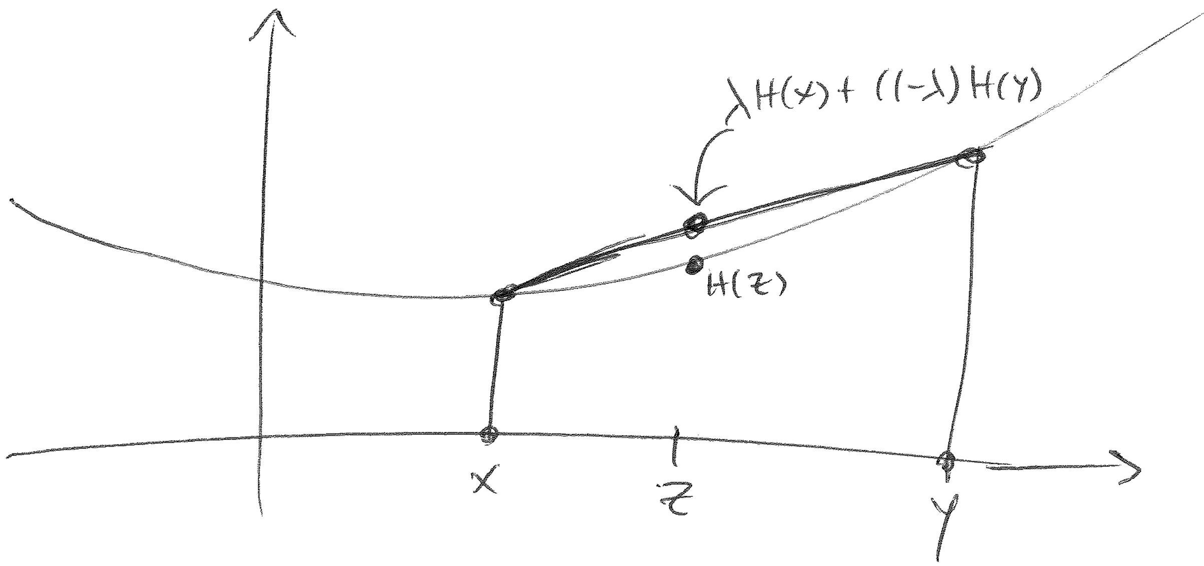
We first need an alternative definition of convexity.

<u>Lemma</u> H is convex ($H'' \geq 0$) if and only if

$$\boxed{H(\lambda x + (1-\lambda)y) \leq \lambda H(x) + (1-\lambda) H(y)}$$

for all $0 \leq \lambda \leq 1$, $x, y \in \mathbb{R}$

See appendix of Calc. of variations notes for proof of the lemma. A useful picture:



$$z = \lambda x + (1-\lambda) y$$

Basically, the lemma is a restatement of the fact that the <u>line</u> between any two points $(x, H(x))$ and $(y, H(y))$ lies above the graph of $H(p)$ when H is <u>convex</u>.

$\boxed{\text{Exercise}}$ If H is convex and superlinear then so is $L = H^*$ (Next homework)

We first need to localize the Hopf-Lax formula. $\boxed{\text{Let } u(x,t) \text{ be given by (H-L).}}$

Lemma: For any $0 < s < t$ and $x \in \mathbb{R}$ we have

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ u(y,s) + (t-s) L\left(\frac{x-y}{t-s}\right) \right\}$$

Basically, we are using $u(y,s)$ as initial data in the Hopf-Lax formula at time $s < t$.

Proof: Let

$$v(x,t) = \min_{y \in \mathbb{R}} \left\{ u(y,s) + (t-s) L\left(\frac{x-y}{t-s}\right) \right\}$$

We need to show $u = v$. Note by (H-L)

$$u(y,s) = \min_{z \in \mathbb{R}} \left\{ f(z) + s L\left(\frac{y-z}{s}\right) \right\}.$$

Plugging this into $v(x,t)$ gives

$$v(x,t) = \min_{y\in\mathbb{R}} \min_{z\in\mathbb{R}} \Big\{ f(z) + \underbrace{(t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right)}_{A} \Big\}$$

Note that

$$A = t\left[\left(\frac{t-s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{s}\right) \right]$$

$$\geq t L\left(\left(\frac{t-s}{t}\right)\left(\frac{x-y}{t-s}\right) + \left(\frac{s}{t}\right)\left(\frac{y-z}{s}\right) \right) \quad (*)$$

$$= t L\left(\frac{x-y+y-z}{t} \right) = t L\left(\frac{x-z}{t}\right)$$

where $(*)$ is by convexity of L with

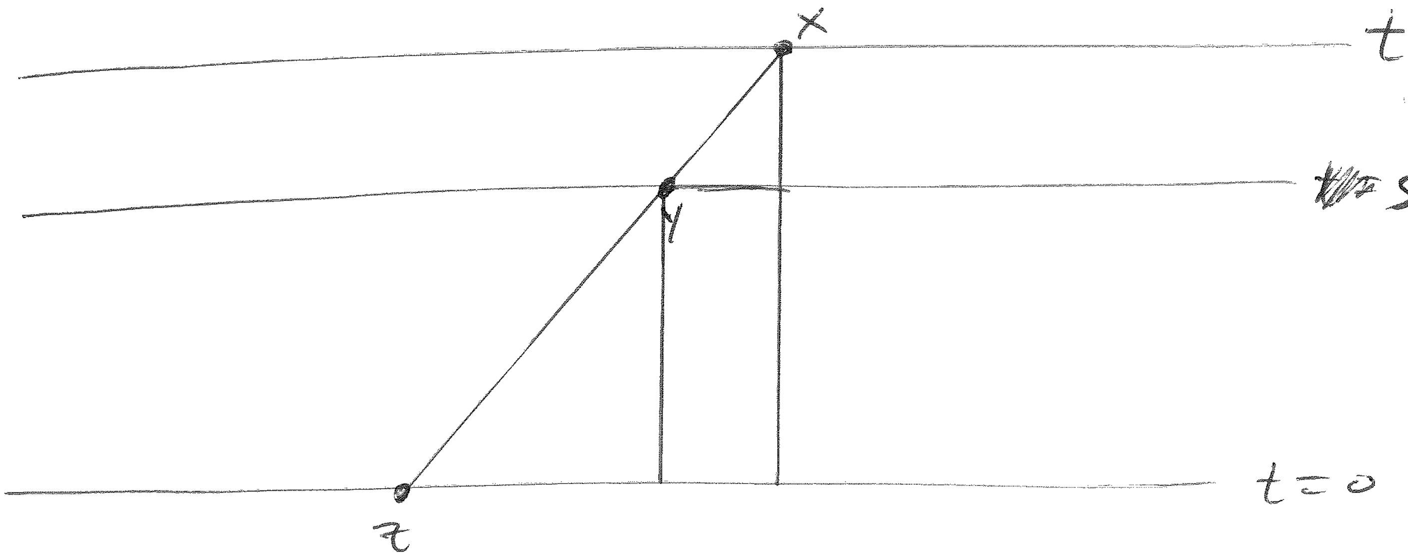
$$\lambda = \frac{t-s}{t}, \quad 1-\lambda = \frac{s}{t}$$

Therefore

$$v(x,t) \geq \min_{z\in\mathbb{R}} \Big\{ f(z) + tL\left(\frac{x-z}{t}\right) \Big\} = u(x,t)$$

Just need to show $v(x,t) \leq u(x,t)$ now.

~~Xxxx~~ For $z \in \mathbb{R}$ choose $y \in \mathbb{R}$ according to the picture below



That is, y is the point at which the line from $(z,0)$ to (x,t) intersects the line $t=s$. By comparing similar triangles

$$\frac{y-z}{s} = \frac{x-z}{t} = \frac{x-y}{t-s}$$

Therefore

$$V(x,t) \le \min_{z \in \mathbb{R}} \left\{ f(z) + (t-s)L\left(\frac{x-z}{t}\right) + sL\left(\frac{x-z}{t}\right) \right\}$$

$$= \min_{z \in \mathbb{R}} \left\{ f(z) + tL\left(\frac{x-z}{t}\right) \right\} = u(x,t)$$

$\blacksquare$

<u>Theorem</u> Assume $u(x,t)$ is smooth (continuously differentiable is enough), where u is given by (H-L). Then

$$u_t + H(u_x) = 0, \quad \begin{matrix} -\infty < x < \infty \\ t > 0 \end{matrix}$$

<u>Remark</u>: In general u is not continuously differentiable, but we'll handle this later.

Proof: This proof is not entirely rigorous, but all steps can be justified rigorously.

Fix (x,t), $x \in \mathbb{R}$, $t > 0$. By previous lemma

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ u(y,s) + (t-s) L\left(\frac{x-y}{t-s}\right) \right\}$$

Hence

$$u(x,t) - \min_{y \in \mathbb{R}} \left\{ u(y,s) + (t-s) L\left(\frac{x-y}{t-s}\right) \right\} = 0$$

or

$$\text{(KEY)} \quad \boxed{\max_{y \in \mathbb{R}} \left\{ u(x,t) - u(y,s) - (t-s) L\left(\frac{x-y}{t-s}\right) \right\} = 0}$$

We now make a Taylor expansion and ignore the error terms

$$u(y,s) = u(x,t) + u_x(x,t)(y-x) + u_t(x,t)(s-t)$$

[See appendix of calc. of variations notes for Taylor expansions]

Therefore

$$u(x,t) = u(y,s) + u_x(x,t)(x-y) + u_t(x,t)(t-s)$$

and we have

$$\max_{y \in \mathbb{R}} \left\{ u_x(x,t)(x-y) + u_t(x,t)(t-s) - (t-s)L\left(\frac{x-y}{t-s}\right) \right\} = 0$$

Divide both sides by $(t-s)$

$$\max_{y \in \mathbb{R}} \left\{ u_x(x,t)\left(\frac{x-y}{t-s}\right) + u_t(x,t) - L\left(\frac{x-y}{t-s}\right) \right\} = 0$$

We can pull $u_t(x,t)$ out of max to get

$$u_t(x,t) + \max_{y \in \mathbb{R}} \left\{ u_x(x,t)\left(\frac{x-y}{t-s}\right) - L\left(\frac{x-y}{t-s}\right) \right\} = 0$$

Now change variables in the sum:

$$q = \frac{x-y}{t-s}.$$

Then we have

$$u_t(x,t) + \max_{q \in \mathbb{R}} \left\{ u_x(x,t)\, q - L(q) \right\} = 0$$

or

$$u_t(x,t) + L^*(u_x(x,t)) = 0$$

Since $L^* = H^{**} = H$ we are done. $\blacksquare$

Since $u(x,t)$ is not continuously differentiable we cannot use Theorem in general. Recall u_x can have discontinuities due to crossing characteristics.

We show here that $u(x,t)$ given by (H-L) is the unique viscosity solution of the Hamilton-Jacobi equation (H).

Theorem: Under reasonable conditions on $f(x)$, $u(x,t)$ is a viscosity solution of (H).

Proof: In the previous theorem (KEY) does not require smoothness of $u(x,t)$. We will start here:

$$\max_{y \in \mathbb{R}} \left\{ u(x,t) - u(y,s) - (t-s) L\left(\frac{x-y}{t-s}\right) \right\} = 0$$

This holds for all $0 < s < t$ and $x \in \mathbb{R}$.

Fix (x,t) and let $\varphi(x,t)$ be a smooth test function such that

$u - \varphi$ has a local maximum at (x,t).

This means

$$u(y,s) - \varphi(y,s) \leq u(x,t) - \varphi(x,t) \qquad (*)$$

for all $|x-y| \leq r$ and $|t-s| \leq r$ for some possibly small $r>0$.

However, we can ~~show~~ actually assume $(*)$ holds globally for all $y \in \mathbb{R}$, $s \in \mathbb{R}$. [This will be on homework].

Hence, by $(*)$ we have

$$u(x,t) - u(y,s) \geq \varphi(x,t) - \varphi(y,s)$$

for all (y,s). Plugging this into (KEY) we have

$$\max_{y \in \mathbb{R}} \left\{ \varphi(x,t) - \varphi(y,s) - (t-s)L\left(\frac{x-y}{t-s}\right) \right\} \leq 0$$

Proceeding as in the previous theorem (since φ is smooth) we get

$$\varphi_t(x,t) + H(\varphi_x(x,t)) \leq 0$$

Hence, $u(x,t)$ is a viscosity subsolution of our Hamilton-Jacobi equation.

The viscosity supersolution property is similar and left for homework. $\boxtimes$

We have shown that the Hopf-Lax formula

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ f(y) + t L\left(\frac{x-y}{t}\right) \right\} \qquad L = H^*$$

is the viscosity solution of the Hamilton-Jacobi equation

$$(H) \quad \begin{cases} u_t + H(u_x) = 0, & -\infty < x < \infty \\ & t > 0 \\ u(x,0) = f(x). \end{cases}$$

when H is convex and superlinear.

Recall: We derived the Hopf-Lax formula as an envelope solution

$$u(x,t) = \min_{y \in \mathbb{R}} \max_{p \in \mathbb{R}} \left\{ f(y) + (x-y)p - H(p)t \right\}$$

If we switch the min/max we get another formula

$$u(x,t) = \max_{p\in\mathbb{R}} \min_{y\in\mathbb{R}} \left\{ f(y) + (x-y)p - tH(p) \right\}$$

$$= \max_{p\in\mathbb{R}} \left\{ xp - tH(p) - \max_{y\in\mathbb{R}} \left\{ yp - f(y) \right\} \right\}$$

$$= \max_{p\in\mathbb{R}} \left\{ xp - tH(p) - f^*(p) \right\}$$

This leads us to the formula

$$\boxed{u(x,t) = \max_{p\in\mathbb{R}} \left\{ xp - tH(p) - f^*(p) \right\}}$$

due to Hopf. This Hopf formula requires f to be convex and superlinear, but places no constraints on H ($H(p)$ can be non-convex).

Back to Scalar Conservation Laws:

If $v(x,t)$ satisfies (H convex, superlinear)

$$(H)\quad \begin{cases} v_t + H(v_x) = 0, & -\infty < x < \infty,\ t>0 \\ v(x,0) = g(x) \end{cases}$$

that is, $v(x,t) = \min_{y \in \mathbb{R}} \left\{ g(y) + tL\left(\frac{x-y}{t}\right) \right\}$

where $L = H^*$, then

$$\boxed{u(x,t) = v_x(x,t)}$$

Solves the conservation law

$$(C\text{-}L)\quad \begin{cases} u_t + \frac{\partial}{\partial x} H(u) = 0, & -\infty < x < \infty,\ t>0. \\ u(x,0) = g'(x)\,! \end{cases}$$

Indeed, we formally differentiate (H) to get

$$\frac{\partial}{\partial x}(v_t) + \frac{\partial}{\partial x} H(v_x) = 0$$

Since $\frac{\partial}{\partial x}(v_t) = v_{tx} = v_{xt} = u_t$

we have

$$u_t + \frac{\partial}{\partial x} H(u) = 0.$$

Thus,

$$\boxed{u(x,t) = \frac{\partial}{\partial x} \min_{y \in \mathbb{R}} \left\{ g(y) + t L\left(\frac{x-y}{t}\right)\right\}} \quad (L\text{-}O)$$

should be a solution formula for the conservation law (C-L). This is called the <u>Lax–Oleinik</u> formula, but usually it is <u>simplified</u> a bit first.

To simplify, let $y(x,t)$ be the minimum in the Hopf-Lax formula. Then

$$\min_{y \in \mathbb{R}} \left\{ g(y) + t L\left(\frac{x-y}{t}\right)\right) = g(y(x,t)) + t L\left(\frac{x-y(x,t)}{t}\right)$$

Note that $y(x,t) = y$ satisfies

$$g'(y) - L'\left(\frac{x-y}{t}\right) = 0$$

Hence $\boxed{g'(y) = L'\left(\frac{x-y}{t}\right)}$

The Lax-Oleinik formula is then

$$u(x,t) = \frac{\partial}{\partial x}\left(g(y(x,t)) + tL\left(\frac{x-y(x,t)}{t}\right)\right)$$

$$= g'(y)\frac{\partial y}{\partial x} + tL'\left(\frac{x-y}{t}\right)\left(\frac{1-\frac{\partial y}{\partial x}}{t}\right)$$

$$= g'(y)\frac{\partial y}{\partial x} + L'\left(\frac{x-y}{t}\right) - L'\left(\frac{x-y}{t}\right)\frac{\partial y}{\partial x}$$

$$= L'\left(\frac{x-y}{t}\right) = L'\left(\frac{x-y(x,t)}{t}\right)$$

As a final step we simplify L':

<u>Note</u>: $L(q) = \max_{p\in\mathbb{R}}\{pq - H(p)\} = H^*(q)$

Therefore $L(q) = pq - H(p)$ where

$$\cancel{p} \quad q - H'(p) = 0$$

or $q = H'(p)$

If $H'' > 0$ then H' is strictly increasing and has an inverse $\boxed{G(p) = (H')^{-1}(p)}$

Thus

$$p = G(q) \quad \text{and}$$

$$L(q) = qG(q) - H(G(q))$$

$$\begin{aligned} L'(q) &= G(q) + qG'(q) - H'(G(q))G'(q) \\ &= G(q) + qG'(q) - qG'(q) \\ &= G(q). \end{aligned}$$

Therefore the Lax-Oleinik formula is

$$\boxed{u(x,t) = G\left(\frac{x - y(x,t)}{t}\right)} \qquad (*)$$

where $G(q) = (H')^{-1}(q)$ and $y(x,t)$ is the minimum in the Hopf-Lax formula

$$\min_{y \in \mathbb{R}} \left\{ f(y) + tL\left(\frac{x-y}{t}\right) \right\}.$$

$(*)$ is the usual form for the Lax-Oleinik formula, and u is the <u>unique</u> <u>entropy</u> solution of the scalar conservation law

$$\begin{cases} u_t + \frac{\partial}{\partial x} H(u) = 0, & -\infty < x < \infty \\ u(x,0) = f'(x) & \end{cases}$$

Notice the Lax-Oleinik formula is

$$H'(u(x,t)) = \frac{x - y(x,t)}{t}$$

or $\quad x = y(x,t) + t H'(u(x,t))$.

This should be familiar from the method of characteristics. In particular $y(x,t)$ is the starting point of the characteristic hitting (x,t).