# Math 5588 Final Exam

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Name:________________________________

**Instructions:**

1. I recommend looking over the problems first and starting with those you feel most comfortable with.
2. Unless otherwise noted, be sure to include explanations to justify each step in your arguments and computations. For example, make sure to check that the hypotheses of any theorems you use are satisfied.
3. All work should be done in the space provided in this exam booklet. Cross out any work you do not wish to be considered. Additional white paper is available if needed.
4. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.
5. If you complete the exam within the last 15 minutes, please remain in your seat until the examination period is over.
6. In the event that it is necessary to leave the room during the exam (e.g., fire alarm), this exam and all your work must remain in the room, face down on your desk.

| Problem | Score |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 | /20 |
| **Total:** | /80 |

1. Find the function $u : [0, 1] \to \mathbb{R}$ that minimizes

$$I(u) = \int_0^1 e^{u(x)} u'(x) + u'(x)^2 \, dx,$$

subject to $u(0) = 0$ and $u(1) = 1$.

2. [10 points] Find the entropy solution of Burger's equation $u_t + uu_x = 0$ with

$$u(x,0) = f(x) = \begin{cases} 1, & \text{if } x < -1 \\ 0, & \text{if } -1 < x < 1 \\ -1, & \text{if } x > 1. \end{cases}$$

Sketch the characteristics and write down $u(x,t)$ explicitly.

3. [10 points] Let $H(p) = e^p$.

(a) [4 points] Show that

$$H^*(q) = \begin{cases} q(\log q - 1), & \text{if } q > 0 \\ 0, & \text{if } q = 0 \\ \infty, & \text{if } q < 0. \end{cases} .$$

(b) [2 points] Is $H$ convex? Is $H$ superlinear?

(c) [4 points] Compute $H^{**}(p)$.

4. [10 points] Solve the wave equation $u_t - \Delta u = 0$ in $n = 3$ dimensions with initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 2x_1^2 - x_2^2 - x_3^2$.

5. [10 points] Let $U \subset \mathbb{R}^n$ and suppose that $u(x,t)$ solves

$$u_t + u + H(\nabla u, \nabla^2 u) = 0 \quad \text{for } x \in U,\ t > 0,$$

with initial condition $u(x,0) = f(x)$ for $x \in U$, and boundary condition $u(x,t) = 0$ for $x \in \partial U$, $t > 0$. If $H$ is (degenerate) elliptic, $H(0,0) = 0$, and $0 \le f(x) \le 1$ for all $x \in U$, show that

$$0 \le u(x,t) \le e^{-t}$$

for all $x \in U$ and $t > 0$. [Hint: Maximum principle]

6. [10 points] Let $U \subset \mathbb{R}^n$ and consider the functional

$$I(u) = \int_U L(x, u, \nabla u, \Delta u)\, dx.$$

Derive the Euler-Lagrange equation for $I$. [Hint: Write $L = L(x, z, p, q)$ where $x \in U$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$, and write $L_z$ and $L_q$ for the partial derivatives in $z$ and $q$, and write $\nabla_p L$ for the gradient in $p$. Take variations of $I$ in directions of compactly supported smooth test functions.]

7. [20 points: 4 for each part] Let $f : \mathbb{R} \to \mathbb{R}$ be a probability density function, that is $f \geq 0$ and

$$\int_{\mathbb{R}} f(x)\, dx = 1.$$

Let us assume that $\int_{\mathbb{R}} x f(x)\, dx = 0$ ($f$ has zero mean) and write

$$\sigma^2 = \int_{\mathbb{R}} x^2 f(x)\, dx$$

for the variance. Also assume $\int_{\mathbb{R}} |x|^3 f(x)\, dx < \infty$.

(a) Show that

$$\sqrt{2\pi}\widehat{f}(k) = 1 - \frac{\sigma^2}{2}k^2 + O(|k|^3).$$

[Hint: Use the definition of $\widehat{f}$ and the Taylor expansion

$$e^{-ikx} = 1 + (-ikx) + \frac{(-ikx)^2}{2} + O(|xk|^3).]$$

(b) If $X_1, \ldots, X_n$ are independent and identically distributed random variables with probability density $f(x)$, then the sum

$$S_n := X_1 + \cdots + X_n$$

has probability density function $g_n(x)$ given by the $n$-fold convolution

$$g_n = \underbrace{f * f * f * \cdots * f}_{n \text{ times}}.$$

Show that

$$\sqrt{2\pi}\,\widehat{g_n}(k) = \left(1 - \frac{\sigma^2}{2}k^2 + O(|k|^3)\right)^n.$$

(c) Let $h_n$ be the probability density function for the normalized sum $S_n/\sqrt{n}$, that is, $h_n(x) = \sqrt{n}\, g_n(\sqrt{n}x)$. Show that

$$\sqrt{2\pi}\,\widehat{h_n}(k) = \left(1 - \frac{\sigma^2}{2n}k^2 + O\left(\tfrac{|k|^3}{n^{3/2}}\right)\right)^n.$$

(d) Show that as $n \to \infty$ we have

$$\widehat{h}(k) := \lim_{n\to\infty} \widehat{h_n}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}k^2}.$$

[Hint: You can neglect the error term $O\left(\frac{|k|^3}{n^{3/2}}\right)$ from part (c). Use the identity

$$\lim_{m\to\infty} (1 - \tfrac{1}{m})^m = \frac{1}{e}.]$$

(e) Find a formula for $h(x)$. [The probability density $h(x)$ is the limiting distribution of the normalized sums $S_n/\sqrt{n}$. You should get a familiar probability density; this is the celebrated Central Limit Theorem in probability.]

Scratch paper

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## Formula Sheet

$$L(u(x),u'(x)) - u'(x)L_p(u(x),u'(x)) = \text{Constant}$$

$$L_z(x,u(x),u'(x)) - \frac{d}{dx}L_p(x,u(x),u'(x)) = 0.$$

$$\nabla I(u) = L_z(x,u,\nabla u) - \operatorname{div}\left(\nabla_p L(x,u,\nabla u)\right) = 0$$

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu_i\,dS.$$

$$\int_U u\Delta v\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}\,dS - \int_U \nabla u\cdot\nabla v\,dx$$

$$\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\,dS$$

$$\int_U \Delta v\,dx = \int_{\partial U}\frac{\partial v}{\partial \nu}\,dS$$

$$\int_U u\operatorname{div}(F)\,dx = \int_{\partial U} u\,F\cdot\nu\,dS - \int_U \nabla u\cdot F\,dx.$$

$$\mathcal{F}(u) = \widehat{u}(k) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u(x)e^{-ik\cdot x}\,dx.$$

$$\mathcal{F}^{-1}(\widehat{u}) = u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{u}(k)e^{ik\cdot x}\,dk.$$

$$\mathcal{F}(u_{x_j}) = ik_j\widehat{u}(k).$$

$$\mathcal{F}(u*v) = (2\pi)^{n/2}\widehat{u}(k)\widehat{v}(k).$$

$$\int_{\mathbb{R}^n}|\widehat{u}(k)|^2\,dk = \int_{\mathbb{R}^n}|u(x)|^2\,dx.$$

$$\mathcal{F}(e^{-|x|^2/2}) = e^{-|k|^2/2}.$$

$$u(x,t) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} tg(y) + f(y) + \nabla f(y)\cdot(y-x)\,dS(y).$$

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B(x,t)}\frac{tf(y) + t\nabla f(y)\cdot(y-x) + t^2 g(y)}{\sqrt{t^2-|x-y|^2}}\,dy.$$

$$k_{ij}^s = -\frac{(\mathbf{x}_i-\mathbf{x}_\ell)\cdot(\mathbf{x}_j-\mathbf{x}_\ell)}{4\text{Area}(T_s)},\quad k_{ii}^s = \frac{|\mathbf{x}_j-\mathbf{x}_\ell|^2}{4\text{Area}(T_s)}.$$

$$\frac{dx}{dt} = \frac{F(u_\ell)-F(u_R)}{u_\ell-u_R},\quad u_\ell > u_R.$$

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{f(y) + tL\left(\frac{x-y}{t}\right)\right\},\quad L = H^*$$

$$H^*(q) = \max_{p\in\mathbb{R}^n}\{p\cdot q - H(p)\}.$$

$$H(p,x) = \min_{a\in A}\{f(x,a)\cdot p + r(x,a)\}.$$