Math 5588 – Homework 12 (Due Tuesday May 2)

1. Suppose $H(p)$ is convex and superlinear. Show that $L = H^*$ is also convex and superlinear. [Hint: For convexity, you need to show that $L(\lambda x+(1-\lambda)y) \leq \lambda L(x)+(1-\lambda)L(y)$ for all x, y and $0 < \lambda < 1$, assuming the same is true for H. Write the left hand above using the definition of L and use the fact that $\max\{f + g\} \leq \max f + \max g$. For superlinearity, show that for every $M > 0$
$$\lim_{|q|\to\infty} \frac{L(q)}{|q|} \geq M.$$
To do this, write
$$L(q) = \max_{p\in\mathbb{R}}\{pq - H(p)\}$$
and choose $p = \frac{q}{|q|}M$ to obtain
$$L(q) \geq M|q| - H(Mq/|q|).$$
Proceed from here with the rest of the proof.]

2. Suppose that $u - \varphi$ has a local maximum at $x \in \mathbb{R}$, which means that for some $r > 0$
$$u(y) - \varphi(y) \leq u(x) - \varphi(x) \quad \text{for all } |x - y| < r.$$
Assume also that u is bounded, that is for some $C > 0$ we have $|u(y)| \leq C$ for all $y \in \mathbb{R}$. Show that there exists another test function ψ such that $\psi'(x) = \varphi'(x)$ and
$$u(y) - \psi(y) \leq u(x) - \psi(x) \quad \text{for all } y \in \mathbb{R}.$$
Thus, we can turn the local max (or min) in the definition of viscosity solutions into a global max (or min) without loss of generality. [Hint: Define
$$\psi(y) = \varphi(y) + K(x - y)^2$$
for a carefully chosen constant K.]

3. Consider the following optimal control problem: The state $x(t) \in \mathbb{R}^n$ obeys the dynamics
$$\dot{x}(s) = \alpha(s) \text{ for } t < s \leq T \text{ and } x(t) = x$$
where $\alpha : [0, T] \to \mathbb{R}^n$ is the control, and the cost is
$$C_{x,t}(\alpha) = \int_t^T |\alpha(s)|^2\, ds + x(T)^2.$$
Find the Hamilton-Jacobi-Bellman equation for the value function
$$u(x, t) = \min_{\alpha:[0,T]\to\mathbb{R}^n} C_{x,t}(\alpha).$$

4. Repeat problem 3 for the dynamics

$$\dot{x}(s) = x(s) + \alpha(s)$$

   where $\alpha : [0, T] \to \mathbb{R}^n$.

5. Repeat problem 3 for the dynamics

$$\dot{x}(s) = \mathbf{1}\alpha(s)$$

   where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and $\alpha : [0, T] \to \mathbb{R}$.