Math 5588 – Homework 7 (Due Thursday March 9)

Recall a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric if $a_{ij} = a_{ji}$, so that $A^T = A$. We say a symmetric matrix A is *positive definite*, written $A \geq 0$, if

$$v^T A v = \sum_{i=1}^n \sum_{j=1}^n a_{ij} v_i v_j \geq 0 \quad \text{for all } v \in \mathbb{R}^n.$$

We write $A \leq B$ whenever $B - A \geq 0$.

1. A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ has n real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, counted with multiplicity. Show that A is positive definite ($A \geq 0$) if and only if $\lambda_1 \geq 0$. [Hint: Recall that

   $$\lambda_1 = \min_{v\neq 0} \frac{v^T A v}{v^T v}.$$

   The right hand side above is called a Rayleigh quotient.]

2. Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix. Show that A is positive definite ($A \geq 0$) if and only if

   $$\det(A) \geq 0 \quad \text{and} \quad \mathrm{Trace}(A) \geq 0.$$

3. (a) Let $A, B \in R^{n\times n}$ be *diagonal* matrices. Show that $A \leq B$ if and only if $a_{ii} \leq b_{ii}$ for all $i \in \{1, \ldots, n\}$.
   (b) Give an example of diagonal matrices $A, B \in \mathbb{R}^{n\times n}$ for which neither $A \leq B$ nor $B \leq A$ hold.

4. Consider a nonlinear PDE in the form

   $$H(x, \nabla u, \nabla^2 u) + F(x, \nabla u) = 0. \tag{1}$$

   Assume that H is linear in $\nabla^2 u$, that is,

   $$\lambda H(x, p, A) + H(x, p, B) = H(x, p, \lambda A + B)$$

   for any λ, A, B. Such a PDE is called *quasilinear*. Show that (1) is elliptic if and only if

   $$A \leq 0 \implies H(x, p, A) \geq 0.$$

5. Consider the quasilinear PDE

   $$-\sum_{i=1}^n \sum_{j=1}^n b_{ij}(x, \nabla u) u_{x_i x_j} + F(x, \nabla u) = 0.$$

   Show that this PDE is elliptic if the matrix $B(x, p) = (b_{ij}(x, p))$ is symmetric and positive definite for all x and p. [Hint: Note that

   $$\sum_{i=1}^n \sum_{j=1}^n b_{ij}(x, \nabla u) u_{x_i x_j} = \mathrm{Trace}(B(x, \nabla u)\nabla^2 u).$$

   Use problem 4. You may want to diagonalize B or $A = \nabla^2 u$ by an orthogonal transformation and use the property $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$.]

6. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix and consider the linear elliptic PDE

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} u_{x_i x_j} = f. \tag{2}$$

   (a) The fundamental solution of (2) satisfies

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} u_{x_i x_j} = \delta.$$

   Let $\widehat{u}(k)$ be the Fourier transform of u, and find $S(k)$ so that

$$S(k)\widehat{u}(k) = 1.$$

   The polynomial $S(k)$ is called the *symbol* of the second order differential operator in (2).

   (b) Show that when $n = 2$, the equation

$$S(k) = 1$$

   describes an ellipse in the $k = (k_1, k_2)$-plane. This is the reason for the terminology *elliptic*. [Aside: For $n \geq 3$, $S(k) = 1$ describes an n-dimensional ellipse.]

7. Consider a functional independent of u, that is

$$I(u) = \int_U L(x, \nabla u)\, dx.$$

Show that the corresponding Euler-Lagrange equation

$$-\text{div}\left(\nabla_p L(x, \nabla u)\right) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(L_{p_i}(x, \nabla u)\Big) = 0$$

is elliptic if L is convex. [Hint: Use the chain rule to expand the divergence and then use problem 5. Recall L is convex if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} L_{p_i p_j}(x, p) v_i v_j \geq 0 \quad \text{for all } v \in \mathbb{R}^n.$$

]