Math 5588 – Homework 8 (Due Thursday March 23)

1. Use an argument based on the maximum principle to show that if $\max f \geq 0$,

   $$u + |\nabla u|^2 - \Delta u = f(x) \quad \text{for } |x| \leq 1$$

   and $u(x) = 0$ for $|x| = 1$ then

   $$u(x) \leq \max f \qquad \text{for all } |x| \leq 1.$$

   What happens if $\max f < 0$?

2. Use the maximum principle to show that if

   $$-\Delta u(x) = f(x) \quad \text{for } |x| < 1$$

   and $u(x) = 0$ for $|x| = 1$ then

   $$u(x) \leq \frac{\max f}{2n}(1 - |x|^2) \qquad \text{for all } |x| \leq 1.$$

   [Hint: Note that $-\Delta v = 2Cn$ for $v(x) = C(1 - |x|^2)$. Choose C so that $-\Delta v \geq -\Delta u$ and use the maximum (or comparison) principle.]

3. Assume $u : \mathbb{R} \to \mathbb{R}$ is a continuous *increasing* function, but not necessarily differentiable. Show that u is a viscosity supersolution of $u'(x) = 0$. [Hint: You need to show that if $u - \varphi$ has a local minimum at x_0 then $\varphi'(x_0) \geq 0$, where φ is a smooth test function. Since $u - \varphi$ has a local minimum at x_0 there exists $r > 0$ such that

   $$u(x) - \varphi(x) \geq u(x_0) - \varphi(x_0) \quad \text{for } |x - x_0| \leq r.$$

   Choose $x = x_0 - h$ for $h > 0$ and send $h \to 0$ to show that $\varphi'(x_0) \geq 0$. Also note that a similar argument shows u is a viscosity subsolution of $-u'(x) = 0$.]

4. Show that $u(x) = -|x|$ is a viscosity solution of $u'(x)^2 - 1 = 0$. [Hint: The only point you need to check is $x = 0$. You need to show that if $u - \varphi$ has a local maximum at $x = 0$ then $-1 \leq \varphi'(x) \leq 1$. Use an argument similar to the previous question. For the supersolution property at $x = 0$, show that there are no smooth test functions φ for which $u - \varphi$ has a local minimum at $x = 0$ by showing that any such function would satisfy $\varphi'(0) \geq 1$ and $\varphi'(0) \leq -1$ (draw a picture of a smooth function touching from below at $x = 0$).]