Math 5588 Midterm I Information

- The midterm will take place on Thursday, February 16, during class.
- The exam will cover everything up to and including the lecture on Thursday, February 2.
- The exam is closed book. No textbooks, notes, or calculators are allowed.
- The formula sheet at the end of these sample problems will be provided in the midterm.
- The exam will have 5 questions. The first 3 will be short, and the last 2 will be longer and slightly more involved. Below are a collection of sample midterm questions for you to practice.

Sample questions

1. Determine whether the following statements are true or false. No justification is required.

   (a) Every solution of the Euler-Lagrange equation for a functional
   $$I(u) = \int_U L(x, u(x), \nabla u(x))\, dx$$
   is a global minimum for I.

   (b) Every function $u : [0,1] \to \mathbb{R}$ is a solution of the Euler-Lagrange equation for the functional
   $$I(u) = \int_0^1 u'(x)\, dx.$$

   (c) Every Euler-Lagrange equation has a solution.

   (d) The solution of the Euler-Lagrange equation is always unique.

2. Find the Euler-Lagrange equation for the functional
   $$I(u) = \int_0^1 \left(u^{(m)}(x)\right)^2\, dx$$
   where $u^{(m)}(x)$ denotes the m^{th} derivative of u.

3. Consider the functional
   $$I(u) = \int_0^1 |u'(x)|\, dx.$$
   Suppose we wish to minimize $I(u)$ over all functions $u : [0,1] \to \mathbb{R}$ satisfying the boundary conditions $u(0) = 0$ and $u(1) = 1$. Find at least two functions that minimize I and find the minimal value of $I(u)$.

4. Find the Euler-Lagrange equation for the functional
   $$I(u) = \int_0^1 u(x)\, dx.$$
   Explain why there are no solutions of this Euler-Lagrange equation.

5. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U \frac{1}{2} \log(1 + |\nabla u|^2)\, dx.$$

Simplify as much as possible.

6. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i} u_{x_j}\, dx.$$

Simplify as much as possible.

7. Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^1 u'(x)^3\, dx$$

subject to the boundary conditions $u(0) = 0$ and $u(1) = 1$.

8. Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^1 \frac{1}{2} u'(x)^2 - u(x)\, dx$$

subject to the boundary conditions $u(0) = 0$ and $u(1) = 1$.

9. Find the minimizer of the constrained optimization problem of maximizing

$$I(u) = \int_0^1 u(x)\, dx$$

subject to $u(0) = u(1) = 0$ and

$$J(u) = \int_0^1 \frac{1}{2} u'(x)^2\, dx = 1.$$

Formula Sheet

$$L(u(x), u'(x)) - u'(x) L_p(u(x), u'(x)) = \text{Constant}$$

$$L_z(x, u(x), u'(x)) - \frac{d}{dx} L_p(x, u(x), u'(x)) = 0.$$

$$\nabla I(u) = L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right) = 0$$

$$\int_U u_{x_i}\, dx = \int_{\partial U} u \nu_i \, dS.$$

$$\int_U u \Delta v \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS - \int_U \nabla u \cdot \nabla v \, dx$$

$$\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS$$

$$\int_U \Delta v \, dx = \int_{\partial U} \frac{\partial v}{\partial \nu} \, dS$$

$$\int_U u \operatorname{div}(F) \, dx = \int_{\partial U} u \, F \cdot \nu \, dS - \int_U \nabla u \cdot F \, dx.$$