# Math 5588 Midterm II

Prof. Jeff Calder
March 30, 2017

Name:________________________________

**Instructions:**

1. I recommend looking over the problems first and starting with those you feel most comfortable with.
2. Unless otherwise noted, be sure to include explanations to justify each step in your arguments and computations. For example, make sure to check that the hypotheses of any theorems you use are satisfied.
3. All work should be done in the space provided in this exam booklet. Cross out any work you do not wish to be considered. Additional white paper is available if needed.
4. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.
5. If you complete the exam within the last 15 minutes, please remain in your seat until the examination period is over.
6. In the event that it is necessary to leave the room during the exam (e.g., fire alarm), this exam and all your work must remain in the room, face down on your desk.

| Problem | Score |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| **Total:** | /50 |

1. [10 points] Do you expect maximum principle to be useful in general for studying Burger's equation
$$u_t + u_x u = 0?$$
Why or why not? Are there any properties of the solution $u$ (or its derivatives) that would make the PDE (degenerate) elliptic and allow the maximum principle to be used?

2. [10 points] Suppose that $u(x)$ solves the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \mathbb{R}^n.$$

Find $\widehat{u}(k)$. [Here, $\Delta^2 u = \Delta\Delta u$.]

3. [10 points] Suppose $u(x,t)$ solves the wave equation

$$u_{tt} - \Delta u + u = 0 \quad \text{in } \mathbb{R}^n$$

with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Solve for $\widehat{u}(k,t)$, where the Fourier transform is taken in $x$ only.

4. [10 points] Let $U \subseteq \mathbb{R}^n$ be open and bounded, and suppose that

$$\Delta u = 0 \quad \text{in } U.$$

Show that

$$\max_{x \in \overline{U}} |\nabla u(x)| = \max_{x \in \partial U} |\nabla u(x)|.$$

[Hint: Write $v(x) = |\nabla u(x)|^2$ and show that $-\Delta v(x) \leq 0$. Then use the maximum principle.]

5. [10 points] Assume $H(p)$ is convex in $p \in \mathbb{R}^n$, and suppose $u(x)$ solves

$$u + H(\nabla u) = 0 \quad \text{in } \mathbb{R}^n,$$

and $u(x) = 0$ for $|x| \geq 1$. Show that for all $i = 1, \ldots, n$

$$u_{x_i x_i}(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

[Hint: Recall that $H$ is convex if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} H_{p_i p_j}(p) v_i v_j \geq 0 \qquad \text{for all } v \in \mathbb{R}^n.$$

Define $v = u_{x_i x_i}$, differentiate the PDE twice in $x_i$, and use convexity of $H$ to show that

$$v + \sum_{j=1}^{n} H_{p_j}(\nabla u) v_{x_j} \leq 0 \qquad \text{in } \mathbb{R}^n.$$

Then use a maximum principle argument to show $v \leq 0$.]

Scratch paper

Scratch paper

Scratch paper

## Formula Sheet

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu_i\,dS.$$

$$\int_U u\Delta v\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}\,dS - \int_U \nabla u\cdot\nabla v\,dx$$

$$\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\,dS$$

$$\int_U \Delta v\,dx = \int_{\partial U} \frac{\partial v}{\partial \nu}\,dS$$

$$\int_U u\,\mathrm{div}(F)\,dx = \int_{\partial U} u\,F\cdot\nu\,dS - \int_U \nabla u\cdot F\,dx.$$

$$\mathcal{F}(u) = \widehat{u}(k) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u(x)e^{-ik\cdot x}\,dx.$$

$$\mathcal{F}^{-1}(\widehat{u}) = u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{u}(k)e^{ik\cdot x}\,dk.$$

$$\mathcal{F}(u_{x_j}) = ik_j\widehat{u}(k).$$

$$\mathcal{F}(u*v) = (2\pi)^{n/2}\widehat{u}(k)\widehat{v}(k).$$

$$\int_{\mathbb{R}^n} |\widehat{u}(k)|^2\,dk = \int_{\mathbb{R}^n} |u(x)|^2\,dx.$$

$$\mathcal{F}(e^{-|x|^2/2}) = e^{-|k|^2/2}.$$