# Math 8385 – Homework 2A (Due Friday November 22)

Let $u \in H^1(U)$ be a weak solution of

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a^{ij} u_{x_i}) = 0 \quad \text{in } U.$$

That is, for every $v \in H_0^1(U)$ we have

$$\int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j}\, dx = 0.$$

Assume the $a^{ij} : U \to \mathbb{R}$ are bounded and measurable, and satisfy the ellipticity condition

$$\theta|\eta|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\eta_i\eta_j \le \Theta|\eta|^2 \qquad (\forall x \in U, \eta \in \mathbb{R}^n),$$

where $0 < \theta \le \Theta$. In this homework, you will show that for $n = 2$ we have $u \in C^{0,\gamma}_{loc}(U)$ for some $\gamma > 0$. This is the interior version of the de Giorgi-Nash-Moser theory.

1. Let $x_0 \in U$ and $r > 0$ such that $B(x_0, 2r) \subset U$.
   (a) Show that there exists a constant $C > 0$, depending only on $\theta$ and $\Theta$, such that

   $$\int_{B(x_0,r)} |Du|^2\, dx \le \frac{C}{r^2} \int_{B(x_0,2r)\setminus B(x_0,r)} |u - a|^2\, dx, \tag{0.1}$$

   where $a$ is any real number. [Hint: Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a smooth cutoff function satisfying $\zeta \equiv 1$ on $B(x_0, r)$, $\zeta \equiv 0$ on $\mathbb{R}^n \setminus B(x_0, 2r)$, $0 \le \zeta \le 1$, and $|D\zeta| \le \frac{2}{r}$. Substitute $v = (u - a)\zeta^2$ into the definition of weak solution.]

   (b) Verify the Poincaré inequality

   $$\int_{B(x_0,2r)\setminus B(x_0,r)} |u - a|^2\, dx \le Cr^2 \int_{B(x_0,2r)\setminus B(x_0,r)} |Du|^2\, dx$$

   holds for

   $$a = ⨍_{B(x_0,2r)\setminus B(x_0,r)} u\, dx.$$

   (c) Combine parts (a) and (b) to deduce

   $$\int_{B(x_0,r)} |Du|^2\, dx \le \frac{C}{C+1} \int_{B(x_0,2r)} |Du|^2\, dx,$$

   where $C > 0$ depends only on $\theta$ and $\Theta$. [Hint: After applying Poincaré's inequality, add $C\int_{B(x_0,r)} |Du|^2\, dx$ to both sides the equation. This is known as the "hole-filling" trick.]

2. Define

$$\varphi(r) := \int_{B(x_0,r)} |Du|^2\, dx.$$

By Part 1, there exists $0 < \eta < 1$, depending only on $\theta$ and $\Theta$, such that

$$\varphi\left(\frac{r}{2}\right) \leq \eta\varphi(r) \qquad \text{for all } 0 < r < r_0, \tag{0.2}$$

where $r_0 = \operatorname{dist}(x_0, \partial U)$.

(a) Show that there exists $0 < \lambda \leq 1$, depending only on $\eta$, such that

$$\varphi(r) \leq \frac{\varphi(r_0)}{\eta}\left(\frac{r}{r_0}\right)^{\lambda} \qquad \text{for all } 0 < r < r_0. \tag{0.3}$$

(b) Use (a) and Poincaré's inequality for a ball to show that

$$⨍_{B(x_0,r)} |u - (u)_{x_0,r}|^2\, dx \leq Cr^{\lambda+2-n} \tag{0.4}$$

for all $0 < r < r_0$, where $C$ depends on $\theta$, $\Theta$, and $r_0 = \operatorname{dist}(x_0, \partial U)$. Recall $(u)_{x_0,r} = ⨍_{B(x_0,r)} u\, dx$.

3. Assume $n = 2$. Use Part 2 to prove that $u \in C^{0,\gamma}_{loc}(U)$ for $\gamma = \lambda/2$. This establishes the local Hölder continuity portion of the de Giorgi-Nash-Moser theory in dimension $n = 2$. **Hint:** Follow the steps below.

(a) Fix $\varepsilon > 0$ and define

$$U_\varepsilon = \{x \in U \,:\, \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Show that for any $x_0 \in U_\varepsilon$ and $0 < s < t < \varepsilon$

$$s^2|(u)_{x_0,s} - (u)_{x_0,t}|^2 \leq C(s^{\lambda+2} + t^{\lambda+2}).$$

(b) Let $x_0 \in U_\varepsilon$, $r < \varepsilon$, and define $r_j = r2^{-j}$ and $a_j = (u)_{x_0,r_j}$. Use part (a) to show that

$$|u(x_0) - (u)_{x_0,r}| \leq \sum_{j=0}^{\infty} |a_{j+1} - a_j| \leq Cr^{\gamma},$$

for almost every $x_0 \in U_\varepsilon$, where $\gamma = \lambda/2$.

(c) Conclude from part (b) that $u \in C(\overline{U_\varepsilon})$ (provided we identify $u$ with its continuous version). [Hint: $(u)_{x,r}$ is a continuous function of $x$ for every $r > 0$.]

(d) Show that $u \in C^{0,\gamma}(\overline{U_\varepsilon})$. [Hint: Let $x, y \in U_\varepsilon$ with $r := |x - y| < \varepsilon$. Write

$$|u(x) - u(y)| \leq |u(x) - (u)_{x,r}| + |(u)_{x,r} - (u)_{y,r}| + |u(y) - (u)_{y,r}|.$$

Estimate the 1st and 3rd terms with part (b). For the second term, mimic the argument used at the end of the proof of Morrey's inequality (Theorem 4 in Evans Section 5.6.2).]