MATH 8385 – HOMEWORK 2B (DUE NOVEMBER 22)

1. Find $L = L(x, z, p)$ so that the PDE

$$-\Delta u + \nabla\varphi \cdot \nabla u = f \quad \text{in } U$$

is the Euler-Lagrange equation associated with the functional $I(u) = \int_U L(x, u, \nabla u)\, dx$. [Hint: Look for a Lagrangian with an exponential term.]

2. Let $u, v \in H_0^1(U)$ be minimizers of the Dirichlet energy

$$I(w) = \int_U |\nabla w|^2, \, dx.$$

Suppose also that $u, v > 0$ within U. Use the hints below to give a new proof that $u = v$ almost everywhere in U. [Hint: Define $w := \left(\frac{u^2+v^2}{2}\right)^{1/2}$, $s := \frac{u^2}{u^2+v^2}$ and $\eta = \frac{u^2+v^2}{2}$. Show that

$$|\nabla w|^2 = \eta \left| s\frac{\nabla u}{u} + (1-s)\frac{\nabla v}{v} \right|^2.$$

Deduce that

$$|\nabla w|^2 \leq \eta \left(s\left|\frac{\nabla u}{u}\right|^2 + (1-s)\left|\frac{\nabla v}{v}\right|^2 \right) = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2,$$

and therefore $\frac{\nabla u}{u} = \frac{\nabla v}{v}$ a.e. in U.]

3. (Pointwise gradient constraint)

   (a) Show there exists a unique minimizer $u \in \mathcal{A}$ of

   $$I(w) := \int_U \frac{1}{2}|\nabla w|^2 - fw\, dx$$

   where

   $$\mathcal{A} = \{w \in H_0^1(U) \,|\, |\nabla w| \leq 1\text{a.e.}\}.$$

   (b) Prove that

   $$\int_U \nabla u \cdot \nabla(w - u)\, dx \geq \int_U f(w - u)\, dx$$

   for all $w \in \mathcal{A}$.