Math 8590 – Homework 1 (Due Friday Sept 21)

Please hand in your solution to 1 problem from those below.

1. Let $u \in \mathrm{USC}(\mathbb{R}^n)$ and define
$$A = \Big\{ x \in \mathbb{R}^n \, : \, \exists \varphi \in C^\infty(\mathbb{R}^n), \ u - \varphi \text{ has a local max at } x \Big\}.$$
Show that A is dense in $\mathbb{R}^n$. [Hint: Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, and consider the maximum of $u - \varphi$ over $B(x_0, 1)$, where $\varphi(x) := \frac{|x-x_0|^2}{\varepsilon}$. Send $\varepsilon \to 0$. You should recall that upper semicontinuous functions assume their maximums over compact sets (why?).]

2. Show that $u(x) = x$ is a viscosity solution of $u' = 1$ on the interval $(0, 1]$, but is *not* a viscosity solution of $u' = 1$ on the interval $[0, 1)$. [Hint: Examine the subsolution condition at $x = 0$. This exercise shows that smooth solutions need not be viscosity solutions at boundary points.]

3. Let $u : (0, 1) \to \mathbb{R}$ be continuous.

   (a) Show that u is nondecreasing on $(0, 1)$ if and only if u is a viscosity solution of $u' \geq 0$ on $(0, 1)$. [Hint: For the hard direction, suppose that $u' \geq 0$ in the viscosity sense on $(0, 1)$, but u is not nondecreasing on $(0, 1)$. Show that there exists $0 < x_1 < x_2 < x_3 < 1$ such that $u(x_3) < u(x_2) < u(x_1)$. Construct a test function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' < 0$ such that φ touches u from below somewhere in the interval (x_1, x_3). Drawing a picture might help.]

   (b) Show that u is convex on $(0, 1)$ if and only if u is a viscosity solution of $-u'' \leq 0$ on $(0, 1)$. Show that in general, convex functions are not viscosity solutions of $u'' \geq 0$. [Hint: The hint for the hard direction is similar to part (a). Suppose that $-u'' \leq 0$ on $(0, 1)$ but u is not convex on $(0, 1)$. Then there exists $0 < x_1 < x_2 < 1$ and $\lambda \in (0, 1)$ such that
$$u(\lambda x_1 + (1 - \lambda)x_2) > \lambda u(x_1) + (1 - \lambda)u(x_2).$$
Construct a test function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi'' < 0$ such that φ touches u from above somewhere in the interval (x_1, x_2).]

4. Let $U \subset \mathbb{R}^n$ be open. Suppose that $u \in C(U)$ satisfies
$$u(x) = \fint_{B(x,\varepsilon)} u \, dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+$$
for every $x \in U$. Note this is an asymptotic version of the mean value property. Show that u is a viscosity solution of
$$-\Delta u = 0 \quad \text{in } U.$$
[Hint: Show that for every $\varphi \in C^\infty(\mathbb{R}^n)$
$$-\Delta \varphi(x) = 2(n+2) \fint_{B(x,\varepsilon)} \frac{\varphi(x) - \varphi(y)}{\varepsilon^2} \, dy + o(1) \quad \text{as } \varepsilon \to 0^+.$$

To do this, write a second order Taylor expansion for φ at x

$$\varphi(y) = \varphi(x) + D\varphi(x) \cdot (y - x) + \frac{1}{2}(y - x)^T D^2\varphi(x)(y - x) + o(|y - x|^2) \quad \text{as } y \to x,$$

and average both sides over the ball $B(x, \varepsilon)$. Then verify the viscosity sub- and super-solution properties directly from the definitions.]