Math 8590 – Homework 1 Solutions

1. Let $u \in \mathrm{USC}(\mathbb{R}^n)$ and define

$$A = \Big\{ x \in \mathbb{R}^n \,:\, \exists \varphi \in C^\infty(\mathbb{R}^n),\ u - \varphi \text{ has a local max at } x \Big\}.$$

Show that A is dense in $\mathbb{R}^n$.

Solution. Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Define

$$\varphi(x) = \frac{1}{\varepsilon}|x - x_0|^2.$$

Since $u - \varphi$ is upper semicontinuous on $B(x_0, 1)$, $u - \varphi$ attains its maximum over the closed ball at some $x_\varepsilon \in B(x_0, 1)$. Since u is upper semicontinuous, u is bounded above on $B(x_0, 1)$. Let $K = \sup_{B(x_0,1)} u$ and note that

$$u(x_\varepsilon) - \frac{1}{\varepsilon}|x_\varepsilon - x_0|^2 \geq u(x_0).$$

Therefore

$$\frac{1}{\varepsilon}|x_\varepsilon - x_0|^2 \leq u(x_\varepsilon) - u(x_0) \leq K - u(x_0),$$

and so $x_\varepsilon \to x_0$ as $\varepsilon \to 0^+$. For $\varepsilon > 0$ sufficiently small, $x_\varepsilon \in B^0(x_0, 1)$, and so $u - \varphi$ has a local maximum at x_ε. Since φ is smooth, we have that $x_\varepsilon \in A$ for $\varepsilon > 0$ sufficiently small, which establishes the density of A. $\square$

2. Show that $u(x) = x$ is a viscosity solution of $u' = 1$ on the interval $(0, 1]$, but is *not* a viscosity solution of $u' = 1$ on the interval $[0, 1)$.

Solution. If $\varphi \in C^\infty(\mathbb{R})$ touches from above or below at $x \in (0, 1)$, then clearly $\varphi'(x) = u'(x) = 1$. Suppose that $u - \varphi$ has a local max at $x = 1$ relative to $(0, 1]$. Then for some $\varepsilon > 0$

$$u(x) - \varphi(x) \leq u(1) - \varphi(1) \quad \text{for all } 1 - \varepsilon \leq x \leq 1.$$

Setting $x = 1 - h$ for $h > 0$ we have

$$\frac{\varphi(1) - \varphi(1-h)}{h} \leq \frac{u(1) - u(1-h)}{h} = 1,$$

and so $\varphi'(1) \leq 1$, which verifies the subsolution property. For the supersolution property, suppose $u - \varphi$ has a local minimum at $x = 1$ relative to $(0, 1]$. Then for some $\varepsilon > 0$

$$u(x) - \varphi(x) \geq u(1) - \varphi(1) \quad \text{for all } 1 - \varepsilon \leq x \leq 1.$$

Setting $x = 1 - h$ for $h > 0$ we have

$$\frac{\varphi(1) - \varphi(1-h)}{h} \geq \frac{u(1) - u(1-h)}{h} = 1,$$

and hence $\varphi'(1) \geq 1$.

To see that u is not a viscosity solution of $u' = 1$ on $[0, 1)$, note that $u - mx = (1 - m)x$ has a local max at $x = 0$ relative to $[0, 1)$ for all $m \geq 1$, which violates the subsolution property. Drawing a picture can be helpful. $\square$

3. Let $u : (0,1) \to \mathbb{R}$ be continuous.

(a) Show that u is nondecreasing on $(0,1)$ if and only if u is a viscosity solution of $u' \geq 0$ on $(0,1)$.

Solution. The proof is split into two steps.

1. Suppose that u is nondecreasing on $(0,1)$. Let $x \in (0,1)$ and $\varphi \in C^\infty(\mathbb{R})$ such that $u - \varphi$ has a local minimum at x. Then for $h > 0$ sufficiently small

$$\varphi(x) - \varphi(x-h) \geq u(x) - u(x-h) \geq 0,$$

as u is nondecreasing. Dividing by $h > 0$ and sending $h \to 0^+$ we find that $\varphi'(x) \geq 0$.

2. Suppose that $u' \geq 0$ in the viscosity sense on $(0,1)$, but u is not nondecreasing on $(0,1)$. Then there exists $0 < x_1 < x_3 < 1$ such that $u(x_1) > u(x_3)$. Since u is continuous, there exists $x_2 \in (x_1, x_3)$ such that

$$u(x_3) < u(x_2) < u(x_1).$$

Define $f : \mathbb{R} \to \mathbb{R}$ by piecing together lines interpolating between $(x_1, u(x_1))$, $(x_2, u(x_2))$, and $(x_3, u(x_3))$ as follows:

$$f(x) = \begin{cases} u(x_2) + m_1(x - x_2), & \text{if } x \leq x_2 \\ u(x_2) + m_2(x - x_2), & \text{if } x > x_2, \end{cases}$$

where

$$m_1 = \frac{u(x_2) - u(x_1)}{x_2 - x_1} \quad \text{and} \quad m_2 = \frac{u(x_3) - u(x_2)}{x_3 - x_2}.$$

Let $\delta = -\max\{m_1, m_2\} > 0$ and note that $f'(x) \leq -\delta$ for all $x \neq x_2$. Let $\varepsilon > 0$ and define

$$f_\varepsilon := \eta_\varepsilon * f,$$

where η_ε is the standard mollifier. Then

$$f'_\varepsilon(x) = (\eta_\varepsilon * f')(x) \leq -\delta < 0 \quad \text{for all } x \in \mathbb{R}.$$

Furthermore, $f_\varepsilon \in C^\infty(\mathbb{R})$ and $f_\varepsilon \to f$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$. Define

$$\varphi(x) = f_\varepsilon(x) - \frac{\delta}{4(x_3 - x_1)}(x - x_2)^2.$$

Then

$$\varphi'_\varepsilon(x) = f'_\varepsilon(x) - \frac{\delta}{2}\left(\frac{x - x_2}{x_3 - x_1}\right) \leq -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0$$

for all $\varepsilon > 0$ and $x \in (x_1, x_3)$. Notice also that

$$u(x_1) - \varphi(x_1) \to \frac{\delta}{4(x_3 - x_1)}(x_1 - x_2)^2,$$

$$u(x_3) - \varphi(x_3) \to \frac{\delta}{4(x_3 - x_1)}(x_3 - x_2)^2,$$

and

$$u(x_2) - \varphi(x_2) \to 0$$

as $\varepsilon \to 0^+$. Therefore, for small enough $\varepsilon > 0$

$$u(x_2) - \varphi(x_2) < \min\{u(x_1) - \varphi(x_1), u(x_3) - \varphi(x_3)\}.$$

It follows that for such an $\varepsilon > 0$, $u - \varphi$ has a local minimum at some $x \in (x_1, x_3)$, at which $\varphi'(x) \leq -\frac{\delta}{2} < 0$, which is a contradiction. □

(b) Show that u is convex on $(0, 1)$ if and only if u is a viscosity solution of $-u'' \leq 0$ on $(0, 1)$. Show that in general, convex functions are not viscosity solutions of $u'' \geq 0$.

Solution. The proof is split into two steps.

1. Suppose that u is convex on $(0, 1)$. Let $x \in (0, 1)$ and $\varphi \in C^\infty(\mathbb{R})$ such that $u - \varphi$ has a local maximum at x. Then

$$\varphi(x) - \varphi(x + h) \leq u(x) - u(x + h)$$

for $|h|$ sufficiently small. Therefore

$$\frac{2\varphi(x) - \varphi(x + h) - \varphi(x - h)}{h^2} \leq \frac{2u(x) - u(x + h) - u(x - h)}{h^2} \leq 0$$

due to the convexity of u. Since the left hand side converges to $-\varphi''(x)$ as $h \to 0$, we deduce that $-\varphi''(x) \leq 0$.

2. Suppose now that u is a viscosity solution of $-u'' \leq 0$ on $(0, 1)$, but u is not convex on $(0, 1)$. Then there exists $0 < x_1 < x_3 < 1$ and $\lambda \in (0, 1)$ such that

$$u(\lambda x_1 + (1 - \lambda)x_3) > \lambda u(x_1) + (1 - \lambda)u(x_3).$$

Let us write $x_2 := \lambda x_1 + (1 - \lambda)x_3$. Define $f : \mathbb{R} \to \mathbb{R}$ to be the line interpolating between $(x_1, u(x_1))$ and $(x_3, u(x_3))$ as follows:

$$f(x) = u(x_1) + m(x - x_1),$$

where

$$m = \frac{u(x_3) - u(x_1)}{x_3 - x_1}.$$

By definition, $u(x_1) = f(x_1)$, $u(x_2) > f(x_2)$, and $u(x_3) = f(x_3)$. Now define

$$\varphi(x) = f(x) - \varepsilon(x - x_2)^2.$$

Note that

$$u(x_2) - \varphi(x_2) = u(x_2) - f(x_2) > 0,$$

$$u(x_1) - \varphi(x_1) = \varepsilon(x_1 - x_2)^2, \quad \text{and} \quad u(x_3) - \varphi(x_3) = \varepsilon(x_3 - x_2)^2.$$

Thus, for $\varepsilon > 0$ sufficiently small

$$u(x_2) - \varphi(x_2) > \max\{u(x_1) - \varphi(x_1), u(x_3) - \varphi(x_3)\}.$$

For such an $\varepsilon > 0$, $u - \varphi$ has a local maximum at some $x \in (x_1, x_3)$ and

$$\varphi''(x) = f''(x) - 2\varepsilon = -2\varepsilon < 0.$$

This contradicts the fact that u is a viscosity solution of $-u'' \le 0$ on $(0, 1)$.

Finally, convex functions are not viscosity solutions of $u'' \ge 0$ because the second derivative of functions that touch from below can be arbitrarily negative, even if u is smooth and convex. For example, let $u(x) = x^2$ and $\varphi(x) = -Cx^2$. Then $u - \varphi$ has a local minimum at $x = 0$ for every $C \ge -1$. Since $\varphi''(0) = -2C$, u is clearly not a viscosity solution of $u'' \ge 0$. Notice that the PDE $u'' \ge 0$ is not even degenerate elliptic (whereas $-u'' \le 0$ is degenerate elliptic). Notice also that these issues are not observed in first order equations. For example, part (a) remains true if $u' \ge 0$ is replaced by $-u' \le 0$. □

4. Let $U \subset \mathbb{R}^n$ be open. Suppose that $u \in C(U)$ satisfies

$$u(x) = \fint_{B(x,\varepsilon)} u\,dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+$$

for every $x \in U$. Show that u is a viscosity solution of

$$-\Delta u = 0 \quad \text{in } U.$$

Solution. Let $\varphi \in C^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then we can expand φ in a second order Taylor series

$$\varphi(y) = \varphi(x) + D\varphi(x) \cdot (y - x) + \frac{1}{2}(y - x)^T D^2\varphi(x)(y - x) + o(|y - x|^2).$$

Take the average of both sides over the ball $B(x, \varepsilon)$ to find that

$$\fint_{B(x,\varepsilon)} \varphi\,dy = \varphi(x) + \frac{1}{2}\fint_{B(x,\varepsilon)} (y - x)^T D^2\varphi(x)(y - x)\,dy + o(\varepsilon^2),$$

where we used the fact that the first order terms in the Taylor expansion are affine, hence harmonic, and their average is $\varphi(x)$ by the mean value property. Set $z = \frac{y-x}{\varepsilon}$ in the second integral to find that

$$\fint_{B(x,\varepsilon)} \varphi(x) - \varphi(y)\,dy + o(\varepsilon^2) = -\frac{\varepsilon^2}{2}\fint_{B(0,1)} z^T D^2\varphi(x) z\,dz. \tag{1}$$

Let us work on the right hand side. We have

$$\fint_{B(0,1)} z^T D^2\varphi(x) z\,dz = \sum_{i,j=1}^{n} \varphi_{x_i x_j}(x) \fint_{B(0,1)} z_i z_j\,dz.$$

Since $z_i z_j$ for $i \neq j$ is harmonic, we can again use the mean value property to obtain

$$\fint_{B(0,1)} z^T D^2\varphi(x) z \, dz = \sum_{i=1}^n \varphi_{x_i x_i}(x) \fint_{B(0,1)} z_i^2 \, dz = \frac{\Delta\varphi(x)}{n} \fint_{B(0,1)} |z|^2 \, dz.$$

Switching to polar coordinates we have

$$\fint_{B(0,1)} |z|^2 \, dz = \frac{1}{\alpha(n)} \int_0^1 \int_{\partial B(0,r)} r^2 \, dS(y) \, dr = \frac{n}{n+2}.$$

Plugging this into (1) we have

$$-\Delta\varphi(x) = 2(n+2) \fint_{B(x,\varepsilon)} \frac{\varphi(x) - \varphi(y)}{\varepsilon^2} \, dy + o(1) \quad \text{as } \varepsilon \to 0^+. \tag{2}$$

Similarly, the assumption on u can be written as

$$\fint_{B(x,\varepsilon)} \frac{u(x) - u(y)}{\varepsilon^2} \, dy = o(1) \quad \text{as } \varepsilon \to 0^+. \tag{3}$$

Now let $x \in \mathbb{R}^n$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x. Then

$$\varphi(x) - \varphi(y) \le u(x) - u(y) \quad \text{for all } y \text{ near } x.$$

Therefore

$$\fint_{B(x,\varepsilon)} \frac{\varphi(x) - \varphi(y)}{\varepsilon^2} \, dy \le \fint_{B(x,\varepsilon)} \frac{u(x) - u(y)}{\varepsilon^2} \, dy = o(1) \quad \text{as } \varepsilon \to 0^+.$$

Combining this with (2) we have

$$-\Delta\varphi(x) \le o(1) \quad \text{as } \varepsilon \to 0^+,$$

and so $-\Delta\varphi(x) \le 0$. Therefore u is a viscosity subsolution of

$$-\Delta u = 0 \quad \text{in } U.$$

Now suppose $u - \varphi$ has a local minimum at x. Then

$$\varphi(x) - \varphi(y) \ge u(x) - u(y) \quad \text{for all } y \text{ near } x,$$

and so

$$\fint_{B(x,\varepsilon)} \frac{\varphi(x) - \varphi(y)}{\varepsilon^2} \, dy \ge \fint_{B(x,\varepsilon)} \frac{u(x) - u(y)}{\varepsilon^2} \, dy = o(1) \quad \text{as } \varepsilon \to 0^+.$$

Combining this with (2) we have

$$-\Delta\varphi(x) \ge o(1) \quad \text{as } \varepsilon \to 0^+,$$

which verifies the supersolution property. □