Math 8590 – Homework 2 Solutions

Please hand in your solution to 1 problem from those below.
Let $U \subset \mathbb{R}^n$ be open.

1. (a) Let $u, v \in \mathrm{USC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2 w, Dw, w, x) \leq 0 \ \text{ in U.} \tag{1}$$

Show that $w(x) := \max\{u(x), v(x)\}$ is a viscosity solution of (1) (i.e., the pointwise maximum of two subsolutions is again a subsolution).

Solution. Let $x \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $w - \varphi$ has a local maximum at x. We can assume that $w(x) = \varphi(x)$ and for some $r > 0$, $w(y) \leq \varphi(y)$ for $y \in B(x, r)$. By definition of w, either $w(x) = u(x)$ or $w(x) = v(x)$. Without loss of generality, assume $w(x) = u(x)$. Then $u(x) = \varphi(x)$ and $u(y) \leq w(y) \leq \varphi(y)$ for $y \in B(x, r)$. Therefore $u - \varphi$ has a local maximum at x and hence

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \leq 0.$$

Since $u(x) = w(x)$, w is a viscosity subsolution of (1). □

(b) Let $u, v \in \mathrm{LSC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2 w, Dw, w, x) \geq 0 \ \text{ in U.} \tag{2}$$

Show that $w(x) := \min\{u(x), v(x)\}$ is a viscosity solution of (1).

Solution. The proof is similar to part (a). □

2. For each $k \in \mathbb{N}$, let $u_k \in C(U)$ be a viscosity solution of

$$H(D^2 u_k, Du_k, u_k, x) = 0 \ \text{ in } U.$$

Suppose that $u_k \to u$ locally uniformly on U (this means $u_k \to u$ uniformly on every $V \subset\subset U$). Show that u is a viscosity solution of

$$H(D^2 u, Du, u, x) = 0 \ \text{ in } U.$$

Thus, viscosity solutions are stable under uniform convergence. (We will see shortly that viscosity solutions are stable under even weaker types of convergence.)

Solution. Let $x \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x. As usual, we can assume $u(x) = \varphi(x)$ and there exists $r > 0$ such that $B(x, r) \subset\subset U$ and $u(y) < \varphi(y)$ for $y \in B(x, r)$, $y \neq x$ (add $|x - y|^2$ to φ to get the strict inequality). Since $u_k \to u$ uniformly on $B(x, r)$, there exists $x_k \to x$ such that $u_k - \varphi$ has a local maximum at x_k for sufficiently large k. We've used this fact several times, so let's give a short proof. Let $x_k \in B(x, r)$ be a point at which the continuous function $u_k - \varphi$ attains its maximum over the closed ball $B(x, r)$. Assume to the contrary that x_k does not converge to x_0. Then there exists a subsequence x_{k_j} and $\delta > 0$ such that $|x_{k_j} - x| > \delta$ for all j. By passing to a further subsequence, if necessary, we may assume that $x_{k_j} \to x_0 \in B(x,r)$, where $|x - x_0| > \delta$. Since

$$u_{k_j}(x_{k_j}) - \varphi(x_{k_j}) \geq u_{k_j}(x) - \varphi(x),$$

and $u_k \to u$ uniformly on $B(x,r)$, we find that $u(x_0) \geq \varphi(x_0)$. Since $u(y) < \varphi(y)$ for $y \neq x$, we have $x_0 = x$, which is a contradiction. Therefore $x_k \to x$ as $k \to \infty$. For sufficiently large k, $x_k \in B^0(x,r)$, so $u_k - \varphi$ has a local max at x_k.

Since $u_k - \varphi$ has a local maximum at x_k

$$H(D^2\varphi(x_k), D\varphi(x_k), u_k(x_k), x_k) \leq 0.$$

Sending $k \to \infty$ and using the continuity of H and uniform convergence of $u_k \to u$ on $B(x,r)$ we have

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \leq 0.$$

Therefore u is a viscosity subsolution. The proof that u is a viscosity supersolution is similar. □

3. Suppose that $H = H(p,x)$ is continuous and $p \mapsto H(p,x)$ is *convex* for any fixed x. Let $u \in C^{0,1}_{loc}(U)$ satisfy

$$\lambda u(x) + H(Du(x), x) \leq 0 \quad \text{for a.e. } x \in U,$$

where $\lambda \geq 0$. Show that u is a viscosity solution of

$$\lambda u + H(Du, x) \leq 0 \ \text{ in } U.$$

Give an example to show that the same result does not hold for supersolutions. [Hint: Mollify u: $u_\varepsilon := \eta_\varepsilon * u$. For $V \subset\subset U$, use Jensen's inequality to show that

$$\lambda u_\varepsilon(x) + H(Du_\varepsilon(x), x) \leq h_\varepsilon(x) \ \text{ for all } x \in V$$

and $\varepsilon > 0$ sufficiently small, where $h_\varepsilon \to 0$ uniformly on V. Then apply an argument similar to problem 2.]

Solution. Let $V \subset\subset U$ and define $u_\varepsilon := \eta_\varepsilon * u$ where η_ε is the standard mollifier. For $\varepsilon < \text{dist}(V, \partial U)$ we have

$$\int_U \eta_\varepsilon(x-y)(\lambda u(y) + H(Du(y), y))\, dy \leq 0,$$

and so

$$\lambda u_\varepsilon(x) + \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) H(Du(y), y))\, dy \leq 0. \tag{3}$$

By Jensen's inequality we have

$$\begin{aligned}\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) H(Du(y), y))\, dy &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) H(Du(y), x))\, dy - h_\varepsilon(x) \\ &\geq H(Du_\varepsilon(x), x) - h_\varepsilon(x),\end{aligned}$$

where

$$h_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)(H(Du(y),x) - H(Du(y),y))\,dy.$$

Therefore

$$\lambda u_\varepsilon(x) + H(Du_\varepsilon(x), x) \le h_\varepsilon(x)$$

for all $x \in V$. Since u is Lipschitz on V and H is continuous (hence uniformly continuous on compact sets), we can show that $h_\varepsilon \to 0$ uniformly on V. We are now ready to show that u is a viscosity subsolution. Let $x \in V$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x. We can assume the local maximum is strict, and so by usual arguments, there exist sequences $\varepsilon_k \to 0^+$ and $x_k \to x$ such that $u_{\varepsilon_k} - \varphi$ has a local maximum at x_k. Since u_{ε_k} is smooth, we have $Du_{\varepsilon_k}(x_k) = D\varphi(x_k)$, and hence

$$\lambda u_{\varepsilon_k}(x_k) + H(D\varphi(x_k), x_k) \le h_{\varepsilon_k}(x_k).$$

Sending $\varepsilon_k \to 0$ and using the uniform convergence $u_\varepsilon \to u$ and $h_\varepsilon \to 0$ on V we have

$$\lambda u(x) + H(D\varphi(x), x) \le 0.$$

Therefore u is a viscosity subsolution. □

4. Let $1 < p < \infty$ and define

$$|x|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}.$$

Assume $U \subset \mathbb{R}^n$ is open, bounded, and path connected with Lipschitz boundary ∂U, and let $f : \overline{U} \to \mathbb{R}$ be continuous and positive. Show that there exists a unique viscosity solution $u \in C(\overline{U})$ of the p-Eikonal equation

$$\text{(P)} \quad \begin{cases} |Du|_p = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Solution. Define

$$u(x) := \inf\{T(x,y) \,:\, y \in \partial U\},$$

where

$$T(x,y) = \inf\left\{\int_0^1 f(\mathbf{w}(t))|\mathbf{w}'(t)|_q\,dt \,:\, \mathbf{w} \in C^1([0,1];\overline{U}),\ \mathbf{w}(0) = x,\ \mathbf{w}(1) = y\right\},$$

and q is the Hölder conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Based on the results from class, $u \in C(\overline{U})$ is locally Lipschitz in U and is a viscosity solution of

$$\begin{cases} H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

where

$$H(r,x) = \sup_{|a|=1}\{-r\cdot a - f(x)|a|_q\}.$$

Let $r \neq 0$ with $r_i \geq 0$ for all i. Set $s_i = -r_i^{p/q}$ and choose $a = \frac{s}{|s|}$ in the definition of H. Then we can compute

$$H(r,x) \geq \frac{1}{|s|}(|r|_p^p - f(x)|r|_p^{p/q}) = |a|_q(|r|_p - f(x)).$$

Since H depends only on the absolute values $|r_i|$, the above holds for all $r \neq 0$ and $a = a(r) \neq 0$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at at x. Then $H(D\varphi(x), x) \leq 0$. If $D\varphi(x) = 0$ then $|D\varphi(x)|_p \leq f(x)$ is trivial, since f is positive. So we may assume $D\varphi(x) \neq 0$. Using $r = D\varphi(x)$ in the above we find that $|D\varphi(x)|_p \leq f(x)$, and so u is a viscosity subsolution of (P).

Let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x. Then $H(D\varphi(x), x)) \geq 0$, and it follows that $D\varphi(x) \neq 0$ (since f is positive). Let $a^* \in \mathbb{R}^n$ with $|a^*| = 1$ such that

$$0 \leq H(D\varphi(x), x) = -D\varphi(x)\cdot a^* - f(x)|a^*|_q.$$

By Hölder's inequality we have

$$0 \leq |a^*|_q(|D\varphi(x)|_p - f(x)).$$

Therefore $|D\varphi(x)|_p \geq f(x)$, and so u is a viscosity supersolution.

Uniqueness follows from the results in class, since H is convex in r and $\varphi \equiv 0$ is a smooth strict subsolution. □