Math 8590 – Homework 4 (Due Friday Nov 2)

Please hand in your solution to 1 problem from those below.

1. Implement both fast sweeping and fast marching in the programming language of your choice for solving the one dimensional eikonal equation

$$|u'(x)| = f(x) \text{ for } x \in (0,1),$$

with boundary conditions $u(0) = u_0$ and $u(1) = u_1$. Experiment with different funtions $f \geq 0$ and different boundary conditions. Are the boundary conditions always attained continuously? Which method is faster? Produce some plots of solutions to hand in, and turn in a working version of your code.

2. Suppose the numerical solutions u_h of our monotone scheme $S_h = 0$ are uniformly Lipschitz continuous, i.e., there exists $C > 0$ such that

$$|u_h(x) - u_h(y)| \leq C|x - y| \quad \text{for all } x, y \in [0,1]^n_h \text{ and } h > 0.$$

This is a stronger form of stability. Prove the Barles-Souganidis convergence theorem from class without assuming **strong uniqueness**. You can assume that ordinary uniqueness holds, that is, there is at most one viscosity solution satisfying the boundary conditions in the usual sense. [Hint: Use the Arzelà-Ascoli Theorem to extract a subsequence u_{h_k} converging uniformly to a continuous function $u \in C([0,1]^n)$. Show that u is the unique viscosity solution, and conclude that the entire sequence must converge uniformly to u.]

3. Suppose that S_h depends only on the forward and backward neighboring grid points in each direction, so that we can write

$$S_h(u, u(x), x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \ldots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x).$$

Let us set $F = F(a_1, \ldots, a_{2n}, z, x)$. You may assume that H and F are smooth.

   (a) Show that S_h is monotone if and only if $F_{a_i} \geq 0$ for all i.

   (b) Show that S_h is consistent if and only if

$$F(p_1, -p_2, \ldots, p_n, -p_n, z, x) = H(p, z, x)$$

   for all $p \in \mathbb{R}^n, z \in \mathbb{R}$ and $x \in [0,1]^n_h$.

   (c) Find a monotone and consistent scheme for the linear PDE

$$a_1 u_{x_1} + \cdots + a_n u_{x_n} = f(x),$$

   where $a_1, \ldots, a_n$ are real numbers. Compare your scheme with the direction of the projected characteristics. [Hint: Your solution should depend on the signs of the a_i.]

(d) Suppose that H is Lipschitz continuous and define

$$a := \sup\{|D_pH(p,z,x)| \,:\, p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in [0,1]^n\}.$$

The Lax-Friedrichs scheme is defined by

$$S_h(u,u(x),x) := H\left(\nabla_h u(x), u(x), x\right) - \frac{ah}{2}\Delta_h u(x),$$

where

$$\nabla_h u(x) := \left(\frac{u(x+he_1)-u(x-he_1)}{2h}, \dots, \frac{u(x+he_n)-u(x-he_n)}{2h}\right),$$

and

$$\Delta_h u(x) := \sum_{i=1}^n \frac{u(x+he_i)-2u(x)+u(x-he_i)}{h^2}.$$

Show that the Lax-Friedrichs scheme is monotone and consistent. [Hint: Rewrite the scheme as a function of the forward and backward differences $\nabla_i^{\pm}u(x)$, as above.]

4. Let $U := B^0(0,1)$ and $\varepsilon > 0$. Consider the nonlocal integral equation

$$(\mathrm{I}_\varepsilon) \quad \begin{cases} (1+c\varepsilon^2)u_\varepsilon(x) - \displaystyle\fint_{B(x,\varepsilon)} u_\varepsilon\,dy = c\varepsilon^2 f(x) & \text{if } x \in U \\ u_\varepsilon(x) = 0 & \text{if } x \in \Gamma_\varepsilon, \end{cases}$$

where $c = \frac{1}{2(n+2)}$, $u_\varepsilon : \Gamma_\varepsilon \cup U \to \mathbb{R}$, $f \in C(\overline{U})$, and

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus U \,:\, \mathrm{dist}(x,\partial U) \le \varepsilon\}.$$

Follow the steps below to show that as $\varepsilon \to 0^+$, u_ε converges uniformly to the viscosity solution u of

$$(\mathrm{P}) \quad \begin{cases} u - \Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

The proof is based on recognizing (I_ε) as a monotone approximation scheme for (P). Unless otherwise specified, any function $u : U \to \mathbb{R}$ is implicitly extended to be identically zero on Γ_ε.

(a) Show that there exists a unique function $u_\varepsilon \in C(\overline{U})$ solving (I_ε). [Hint: Show that the mapping $T : C(\overline{U}) \to C(\overline{U})$ defined by

$$T[u](x) := \frac{1}{1+c\varepsilon^2}\fint_{B(x,\varepsilon)} u\,dy + \frac{c\varepsilon^2}{1+c\varepsilon^2}f(x)$$

is a contraction mapping. Use the usual norm $\|u\| := \max_{\overline{U}} |u|$ on $C(\overline{U})$. Then appeal to Banach's fixed point theorem.]

(b) Define $S_\varepsilon : L^\infty(U \cup \Gamma_\varepsilon) \times \mathbb{R} \times U \to \mathbb{R}$ by

$$S_\varepsilon(u,t,x) := (1 + c\varepsilon^2)t - \fint_{B(x,\varepsilon)} u \, dy.$$

Show that S_ε is monotone, i.e., for all $t \in \mathbb{R}$, $x \in U$, and $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$

$$u \leq v \text{ on } B(x,\varepsilon) \implies S_\varepsilon(u,t,x) \geq S_\varepsilon(v,t,x).$$

(c) Show that the following comparison principle holds: Let $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$ such that $u|_{\overline{U}}, v|_{\overline{U}} \in C(\overline{U})$. If $u \leq v$ on Γ_ε and $S_\varepsilon(u, u(x), x) \leq S_\varepsilon(v, v(x), x)$ at all $x \in U$, then $u \leq v$ on U.

(d) Use the comparison principle to show that there exists $C > 0$ such that

$$|u_\varepsilon(x)| \leq C(1 + 3\varepsilon - |x|^2),$$

for all $x \in U$ and $0 < \varepsilon \leq 1$, where C depends only on $\|f\| = \max_{\overline{U}} |f|$. [Hint: Compare against $v(x) := C(1 + 3\varepsilon - |x|^2)$ and $-v$, and adjust the constant C appropriately.]

(e) Use the method of weak upper and lower limits to show that $u_\varepsilon \to u$ uniformly on $\overline{U}$, where u is the viscosity solution of (P). You may assume a comparison principle holds for (P) for semicontinuous viscosity solutions. That is, if $u \in \mathrm{USC}(\overline{U})$ is a viscosity subsolution of (P) and $v \in \mathrm{LSC}(\overline{U})$ is a viscosity supersolution, and $u \leq v$ on ∂U, then $u \leq v$ in U. [Hint: You will find the identity in the hint from HW1 Problem 4 useful.]