Math 8590: Viscosity Solutions
Bellman Equation

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Shortest paths

Consider the following calculus of variations problem:

$$T(x,y) = \inf\left\{ I[\mathbf{w}] \,:\, \mathbf{w}\in C^1([0,1];\overline{U}),\ \mathbf{w}(0)=x,\ \text{and } \mathbf{w}(1)=y \right\}, \tag{1}$$

where

$$I[\mathbf{w}] := \int_0^1 L(\mathbf{w}'(t),\mathbf{w}(t))\,dt. \tag{2}$$

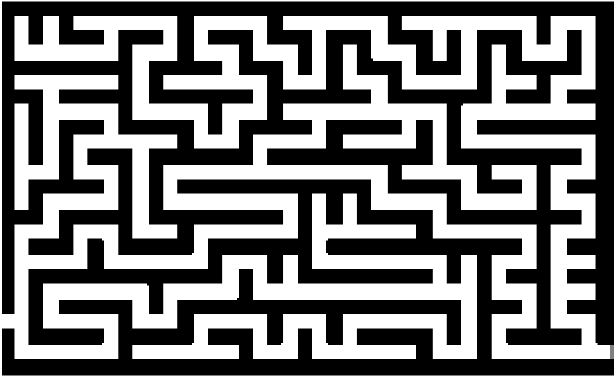
We assume that $L : \mathbb{R}^n \times \overline{U} \to \mathbb{R}$ is continuous, $L(p,x) > 0$ for $p \neq 0$ and

$$p \mapsto L(p,x) \text{ is positively 1-homogeneous.} \tag{3}$$

For $g : \partial U \to \mathbb{R}$ we define the *value function*

$$u(x) = \inf_{y\in\partial U} \left\{ g(y) + T(x,y) \right\}. \tag{4}$$

Shortest paths: Maze navigation



Shortest paths

Proposition 1. *For any* $x, y \in \overline{U}$ *such that the line segment between* x *and* y *belongs to* $\overline{U}$ *we have*

$$T(x,y) \leq K|x-y|, \tag{5}$$

where $K = \sup_{x \in \overline{U}, |p|=1} L(p,x)$.

Lemma 1. *For all* $x, y, z \in \overline{U}$ *we have*

$$T(x,z) \leq T(x,y) + T(y,z). \tag{6}$$

Shortest paths: Dynamic programming principle

Lemma 2. *For every $B(x, r) \subset U$ we have*

$$u(x) = \inf_{y \in \partial B(x,r)} \{u(y) + T(x, y)\}. \tag{7}$$

Shortest paths: Regularity

Here, we need a *compatibility condition* on the boundary values $g : \partial U \to \mathbb{R}$.

$$g(x) - g(y) \leq T(x, y) \quad \text{for all } x, y \in \partial U \tag{8}$$

Lemma 3. *The value function u is locally Lipschitz continuous in U and assumes the boundary values $u = g$ on ∂U, in the sense that for all $x \in \partial U$*

$$\lim_{\substack{y \to x \\ y \in U}} u(y) = g(x).$$

Shortest paths: Bellman equation

We can now characterize u as the viscosity solution of a Hamilton-Jacobi equation. We define

$$H(p,x) = \sup_{|a|=1} \left\{-p\cdot a - L(a,x)\right\}. \tag{9}$$

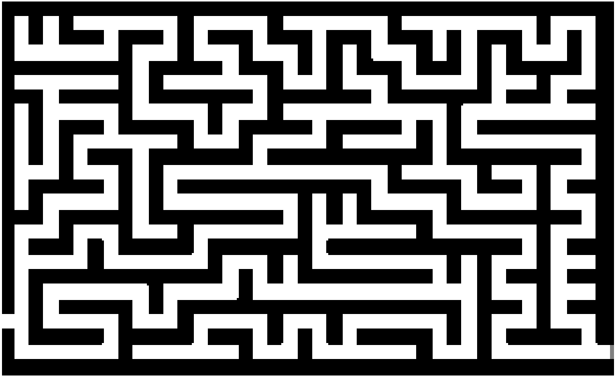
Lemma 4. *H is convex in p and satisfies*

$$H(p,y) - H(p,x) \le \omega(|x-y|(1+|p|)). \tag{10}$$

Theorem 1. *The value function u is the unique viscosity solution of the Bellman equation*

$$\begin{cases} H(Du,x) = 0 & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{11}$$

Shortest paths: Maze navigation



Shortest paths: Dynamic programming

Recall the dynamic programming principle.

$$u(x) = \inf_{y \in \partial B(x,r)} \{u(y) + T(x,y)\}. \tag{12}$$

Once we compute the value function u, we can compute nearly optimal paths via dynamic programming

$$x^{k+1} = \operatorname*{arg\,min}_{x \in \partial B(x^k, \varepsilon)} \left\{ u(x) + T(x^k, x) \right\}. \tag{13}$$