Math 8590: Viscosity Solutions
Boundary conditions in viscosity sense

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Vanishing viscosity

Exercise 1. Consider the ordinary differential equation

$$u_\varepsilon'(x) - \varepsilon u_\varepsilon''(x) = 1, \quad u_\varepsilon(0) = u_\varepsilon(1) = 0.$$

Find explicitly the solution u_ε and sketch its graph. Show that $u_\varepsilon(x) \to x$ pointwise on $[0, 1)$ as $\varepsilon \to 0$.

Vanishing viscosity & boundary conditions

Let u_ε be a smooth solution of

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 \quad \text{in } U, \tag{1}$$

and assume that $u_\varepsilon \leq g$ on ∂U. Consider the weak upper limit

$$\overline{u}(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y).$$

Let $x \in \partial U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a strict local max at x. Show that

$$\min\{H(D\varphi(x), \overline{u}(x), x), \overline{u}(x) - g(x)\} \leq 0.$$

Vanishing viscosity & boundary conditions

Let u_ε be a smooth solution of

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 \quad \text{in } U, \tag{2}$$

and assume that $u_\varepsilon \le g$ on ∂U. Consider the weak upper limit

$$\overline{u}(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y).$$

Let $x \in \partial U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a strict local max at x. Show that

$$\min\{H(D\varphi(x), \overline{u}(x), x), \overline{u}(x) - g(x)\} \le 0.$$

We can make the same argument with the weak lower limit $\underline{u}$ to find that when $\underline{u} - \varphi$ has a local minimum at $x \in \partial U$ we have

$$\max\{H(D\varphi(x), \underline{u}(x), x), \underline{u}(x) - g(x)\} \ge 0,$$

provided $u_\varepsilon \ge g$ on ∂U.

Boundary conditions in the viscosity sense

$$\begin{cases} H(Du, u, x) = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{3}$$

This motivates the following definitions.

Definition 1. We say $u \in \mathrm{USC}(\overline{U})$ is a *viscosity subsolution* of (3) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x with respect to $\overline{U}$

$$\begin{cases} H(D\varphi(x), u(x), x) \leq 0, & \text{if } x \in U \\ \min\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \leq 0 & \text{if } x \in \partial U. \end{cases}$$

Likewise, we say that $u \in \mathrm{LSC}(\overline{U})$ is a *viscosity supersolution* of (3) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x with respect to $\overline{U}$

$$\begin{cases} H(D\varphi(x), u(x), x) \geq 0, & \text{if } x \in U \\ \max\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \geq 0 & \text{if } x \in \partial U. \end{cases}$$

Finally, we say that u is a *viscosity solution* of (3) if u is both a viscosity sub- and supersolution. In this case, we say that the boundary conditions in (3) hold in the *viscosity sense*

Exercise 2. Show that $u(x) = x$ is a viscosity solution of

$$u'(x) = 1, \quad u(0) = u(1) = 0,$$

on the interval $U = (0, 1)$ in the sense of Definition 1.

Comparison principle

We assume the usual monotonicity and regularity conditions on H hold. In addition we assume

$$|H(p,z,x) - H(q,z,x)| \le \omega_1(|p-q|), \tag{4}$$

where ω_1 is a modulus of continuity.

Theorem 1. *Let $U \subset \mathbb{R}^n$ be open and suppose $\partial U = \Gamma_1 \cup \Gamma_2$ where Γ_1 is relatively open and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Let $u \in USC(\overline{U})$ be a bounded viscosity solution of*

$$H(Du,u,x) + \varepsilon \le 0 \text{ on } U \cup \Gamma_1,$$

and let $v \in LSC(\overline{U})$ be a bounded viscosity solution of

$$H(Dv,v,x) \ge 0 \text{ on } U \cup \Gamma_1.$$

If $u \le v$ on Γ_2 then $u \le v$ on U.

Time-dependent equations on $\mathbb{R}^n$

As an application we will prove a comparison principle for the time-dependent Hamilton-Jacobi equation

$$
(5) \qquad \begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}
$$

We assume H satisfies the usual monotonicity and regularity, as well as (4).

Theorem 2. *Let $u \in USC(\mathbb{R}^n \times [0,T])$ be a bounded viscosity subsolution of (5), and let $v \in LSC(\mathbb{R}^n \times [0,T])$ be a bounded viscosity supersolution of (5). Then $u \le v$ on $\mathbb{R}^n \times [0,T]$.*

We say u is a subsolution of (5), we mean that u is a solution of $u_t + H \le 0$ in $\mathbb{R}^n \times (0,T)$ *and* $u \le g$ at $t = 0$. Likewise, a supersolution is assumed to satisfy $v \ge g$ at $t = 0$, hence $u \le v$ at $t = 0$.

Time-dependent equations on $\mathbb{R}^n$

Continuous dependence on initial data.

Corollary 1. *Let $u, v \in C(\mathbb{R}^n \times [0,T])$ be bounded, and assume that $w := u$ and $w := v$ are viscosity solutions of*

$$w_t + H(Dw, x) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Then

$$\sup_{\mathbb{R}^n \times [0,T]} |u - v| \leq \sup_{x \in \mathbb{R}^n} |u(x,0) - v(x,0)|.$$

The Hopf-Lax Formula

In the case that $H = H(p)$ and H is convex and superlinear we have the Hopf-Lax formula

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\},$$

where

$$L(v) = \sup_{p \in \mathbb{R}^n} \{p \cdot v - H(p)\}$$

is the Legendre transform of H.

Exercise 3. Prove that the Hopf-Lax formula gives the unique viscosity solution of (6).

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{6}$$