Math 8590: Viscosity Solutions
Comparison Principle

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Partial order on symmetric matrices

Definition 1. Given A, B real symmetric matrices, we say $A \leq B$ if

$$\forall v \in \mathbb{R}^n \quad v \cdot Av \leq v \cdot Bv.$$

Properties:

- Transitive: $A \leq B$ and $B \leq C$ implies $A \leq C$.
- Reflexive: $A \leq A$.
- Anitsymmetric: $A \leq B$ and $B \leq A$ implies $A = B$.
- If $A \leq B$ then $A + C \leq B + C$.

Partial order on symmetric matrices

Definition 2. Given A, B real symmetric matrices, we say $A \leq B$ if

$$\forall v \in \mathbb{R}^n \quad v \cdot Av \leq v \cdot Bv.$$

Exercise 1. If A, B are diagonal, then $A \leq B$ if and only if $a_{ii} \leq b_{ii}$ for all i. In particular, there exist A, B for which both $A \leq B$ and $B \leq A$ do not hold.

Exercise 2. If $u \in C^2(\mathbb{R}^n)$ has a local maximum at x_0 then $D^2u(x_0) \leq 0$.

In particular, if $u - v$ has a local maximum at x_0 then $D^2u(x_0) \leq D^2v(x_0)$.

The maximum principle (in more generality)

Suppose that $u \in C^2(U) \cap C(\overline{U})$ is a solution of

$$H(D^2u, Du, u, x) \leq 0 \quad \text{in } U.$$

and $v \in C^2(U) \cap C(\overline{U})$ is a solution of

$$H(D^2v, Dv, v, x) > 0 \quad \text{in } U.$$

Question: When can we conclude that the maximum principle

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v)$$

holds?

The maximum principle (in more generality)

The maximum principle requires *monotonicity*

$$(1) \qquad H(X,p,r,x) \le H(X,p,s,x) \quad \text{whenever } r \le s,$$

and *degenerate ellipticity*

$$(2) \qquad H(X,p,z,x) \ge H(Y,p,z,x) \quad \text{whenever } X \le Y.$$

Examples of degenerate elliptic PDE

Degenerate ellipticity:

(3) $$H(X, p, z, x) \geq H(Y, p, z, x) \quad \text{whenever } X \leq Y.$$

Examples:

- Every first order equation.
- Nondivergence form quasilinear equations when $A = (a_{ij}) \geq 0$:

$$-\sum_{i,j=1}^{n} a_{ij}(Du, x) u_{x_i x_j} = 0.$$

- Euler-Lagrange equation for convex variational problems $\min \int L(Du, x)\, dx$:

$$-\mathrm{div}\,(\nabla_p L(Du, x)) = \sum_{i,j=1}^{n} L_{p_i p_j}(Du, x) u_{x_i x_j} = 0.$$

- Mean curvature motion.

$$u_t - |Du|\mathrm{div}\left(\frac{Du}{|Du|}\right) = 0.$$

- Decreasing functions of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of D^2u (e.g., convex Monge-Ampere equation) since we have the min-max formula for symmetric matrices

$$\lambda_k = \min_{\dim(L)=k} \max_{v \in L} \frac{v \cdot Av}{|v|^2}.$$

Comparison for viscosity solutions (1st order)

Let $U \subset \mathbb{R}^n$ be open and bounded, and assume

$$H(p,r,x) \le H(p,s,x) \quad \text{whenever } r \le s, \tag{4}$$

and

$$H(p,z,y) - H(p,z,x) \le \omega(|x-y|(1+|p|)) \tag{5}$$

for all $x, y \in U$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$, where ω is a modulus of continuity (i.e., ω is nonnegative, $\omega(0) = 0$ and ω is continuous at 0).

Theorem 1 (Comparison with strict subsolution)**.** *Let $u \in USC(\overline{U})$ be a viscosity solution of $H \le -\varepsilon$ in U and let $v \in LSC(\overline{U})$ be a viscosity solution of $H \ge 0$ in U. If $u \le v$ on ∂U then $u \le v$ on U.*

Comparison for viscosity solutions (1st order)

Theorem 2 (Comparison with strict subsolution)**.** *Let* $u \in USC(\overline{U})$ *be a viscosity solution of* $H \leq -\varepsilon$ *in* U *and let* $v \in LSC(\overline{U})$ *be a viscosity solution of* $H \geq 0$ *in* U*. If* $u \leq v$ *on* ∂U *then* $u \leq v$ *on* U*.*

Proof. Assume, by way of contradiction, that $u \leq v$ on ∂U and

$$\max_{\overline{U}}(u - v) > 0.$$

For $\alpha > 0$ define the auxilliary function

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2.$$

Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U}\times\overline{U}} \Phi.$$

. . . □

Comparison for viscosity solutions (1st order)

Why does the proof fail for second order equations?

Comparison for viscosity solutions (1st order)

Corollary 1 (Comparison principle)**.** *Let* $u \in USC(\overline{U})$ *be a viscosity solution of* $H \leq 0$ *in* U *and let* $v \in LSC(\overline{U})$ *be a viscosity solution of* $H \geq 0$ *in* U*. Suppose there exists a sequence* $u_k \in USC(\overline{U})$ *such that*

$$u_k \to u \text{ pointwise on } U,$$

$$u_k \leq v \text{ on } \partial U,$$

and each u_k *satisfies in the viscosity sense*

$$H(Du_k, u_k, x) \leq -\frac{1}{k} \text{ in } U$$

Then $u \leq v$ *on* U*.*

Comparison for viscosity solutions (1st order)

Comparison holds when:

1. There exists $\gamma > 0$ such that

$$H(p, z+h, x) - H(p, z, x) \geq \gamma h \quad (h > 0). \tag{6}$$

2. There exists $\gamma > 0$ and $i \in \{1, \ldots, n\}$ such that

$$H(p+he_i, z, x) - H(p, z, x) \geq \gamma h \quad (h > 0). \tag{7}$$

3. If $H(p, z, x) = H(p, x)$, suppose $p \mapsto H(p, x)$ is convex, and there exists $\varphi \in C^\infty(\overline{U})$ such that

$$H(D\varphi(x), x) + \gamma \leq 0 \quad \text{in } U$$

where $\gamma > 0$.