Math 8590: Viscosity Solutions
Motivation and definitions

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Example 1 (Distance function)**.** Let Γ be a closed subset of $\mathbb{R}^n$ and let $u : \mathbb{R}^n \to [0, \infty)$ be the *distance function to* Γ, defined by

$$u(x) = \operatorname{dist}(x, \Gamma) := \min_{y \in \Gamma} |x - y|. \tag{1}$$

Last time we saw that $|Du(x)| = 1$ at any point where u is differentiable.

Exercise 1. Show that there are in general infinitely many Lipschitz almost everywhere solutions of the eikonal equation $|u'(x)| = 1$ on $(-1, 1)$ with $u(-1) = u(1) = 0$.

Vanishing viscosity

We can regularize the equation by adding *viscosity*:

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0.$$

This equation is semilinear and uniformly elliptic. In general, it admits a unique classical solution u_ε subject to some appropriate boundary conditions.

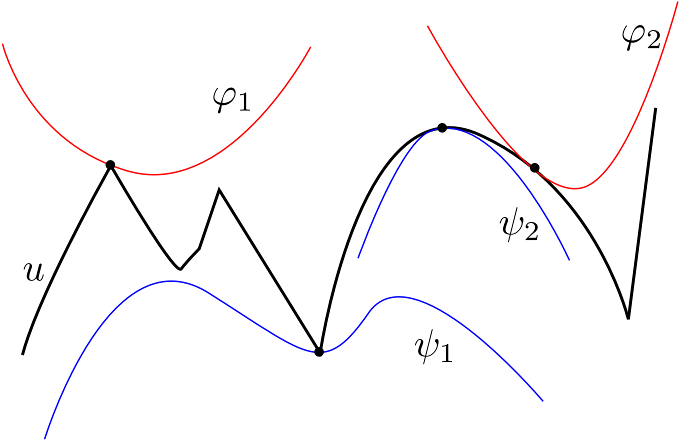
Exercise 2. For $\varepsilon > 0$ consider the ODE

$$|u_\varepsilon'(x)| - \varepsilon u_\varepsilon''(x) = 1 \quad \text{on } (-1,1) \tag{2}$$

with boundary condition $u(-1) = 0 = u(1)$. Compute u_ε and show that

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) = 1 - |x|. \tag{3}$$

Touching with smooth test functions



$u - \varphi_i$ have local maxima (touching from above).

$u - \psi_i$ have local minima (touching from below).

Vanishing viscosity

Consider again the viscous regularized equation

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 \quad \text{on } \mathbb{R}^n.$$

Assume $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$, and let $\varphi \in C^\infty(\mathbb{R}^n)$.

- If $u - \varphi$ has a local maximum at x_0 then

$$H(D\varphi(x_0), u(x_0), x_0) \leq 0.$$

- If $u - \varphi$ has a local minimum at x_0 then

$$H(D\varphi(x_0), u(x_0), x_0) \geq 0.$$

Important property of touching

If $u - \varphi$ has a local maximum at x, then

$$u(y) - u(x) \leq \varphi(y) - \varphi(x) \quad \text{for } y \text{ near } x.$$

If $u - \varphi$ has a local minimum at x, then

$$u(y) - u(x) \geq \varphi(y) - \varphi(x) \quad \text{for } y \text{ near } x.$$

Dynamic programming

Recall the distance function

$$u(x) = \mathrm{dist}(x, \Gamma) := \min_{y \in \Gamma} |x - y|. \tag{4}$$

Last time we saw that u satisfies the **dynamic programming principle**

$$\max_{y \in \partial B(x,r)} \{u(x) - u(y) - r\} = 0. \tag{5}$$

If $u - \varphi$ has a local maximum at x, then

$$\max_{y \in \partial B(x,r)} \{\varphi(x) - \varphi(y) - r\} \leq 0,$$

and so $|D\varphi(x)| \leq 1$.

Similarly, if $u - \varphi$ has a local minimum at x, then $|D\varphi(x)| \geq 1$.

The maximum principle

Suppose that $u \in C^1(U) \cap C(\overline{U})$ is a solution of

$$H(Du, x) = 0 \quad \text{in } U.$$

If $\varphi \in C^\infty(\mathbb{R}^n)$ is any function satisfying

$$H(D\varphi, x) > 0 \quad \text{in } U,$$

then we have the maximum principle

$$\max_{\overline{U}}(u - \varphi) = \max_{\partial U}(u - \varphi),$$

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$$\max_{\overline{U}}(u - \varphi) = \max_{\partial U}(u - \varphi),$$

In fact, we also have

$$u \leq \varphi \text{ on } \partial U \implies u < \varphi \text{ in } U.$$

The maximum principle

Let $V \subset\subset U$, $\varphi \in C^\infty(\mathbb{R}^n)$.

Subsolution property:

$$\text{If } u \leq \varphi \text{ on } \partial V \text{ and } H(D\varphi, x) > 0 \text{ then } u < \varphi \text{ in } V. \tag{6}$$

Supersolution property:

$$\text{If } u \geq \varphi \text{ on } \partial V \text{ and } H(D\varphi, x) < 0 \text{ then } u > \varphi \text{ in } V. \tag{7}$$

If $u \in C(U)$ satisfies (6) for all $V \subset\subset U$ and all $\varphi \in C^\infty(\mathbb{R}^n)$ then

$$u - \varphi \text{ local max at } x_0 \implies H(D\varphi(x_0), u(x_0), x_0) \leq 0.$$

Same for (6) and local min.

Definitions

We consider the general second order nonlinear PDE

$$H(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}, \tag{8}$$

where H is continuous and $\mathcal{O} \subset \mathbb{R}^n$.

Definition 1. We say that a function $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* at $x \in \mathcal{O}$ provided

$$\limsup_{\mathcal{O} \ni y \to x} u(y) \leq u(x).$$

Similarly, a function $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ is *lower semicontinuous* at $x \in \mathcal{O}$ provided

$$\liminf_{\mathcal{O} \ni y \to x} u(y) \geq u(x).$$

Let $\mathrm{USC}(\mathcal{O})$ (resp. $\mathrm{LSC}(\mathcal{O})$) denote the collection of functions that are upper (resp. lower) semicontinuous at all points in $\mathcal{O}$.

Definition of viscosity subsolution

We consider the general second order nonlinear PDE

$$H(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}, \tag{9}$$

Definition 2. We say that $u \in \mathrm{USC}(\mathcal{O})$ is a *viscosity subsolution* of (9) if for every $x \in \mathcal{O}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x with respect to $\mathcal{O}$

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \leq 0.$$

We will often say that $u \in \mathrm{USC}(\mathcal{O})$ is a viscosity solution of $H \leq 0$ in $\mathcal{O}$ when u is a viscosity subsolution of (9).

Definition of viscosity supersolution

We consider the general second order nonlinear PDE

$$H(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}, \tag{10}$$

Definition 3. We say that $u \in \mathrm{LSC}(\mathcal{O})$ is a *viscosity supersolution* of (10) if for every $x \in \mathcal{O}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x with respect to $\mathcal{O}$

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \geq 0.$$

We will often say that $u \in \mathrm{LSC}(\mathcal{O})$ is a viscosity solution of $H \geq 0$ in $\mathcal{O}$ when u is a viscosity supersolution of (10).

Definition 4. We say u is *viscosity solution* of (10) if u is both a viscosity subsolution and a viscosity supersolution.

Classical solutions are viscosity solutions

We can relax the condition $\varphi \in C^\infty(\mathbb{R}^n)$ to $\varphi \in C^2(\mathbb{R}^n)$ for second order equations and $\varphi \in C^1(\mathbb{R}^n)$ for first order equations, provided H is continuous.

Comparison against smooth sub/super solutions

Theorem 1. *Let $U \subset \mathbb{R}^n$ be open and bounded and suppose $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies*

$$H(D^2\varphi, D\varphi, x) > 0 \text{ in } U \tag{11}$$

and let $u \in USC(\overline{U})$ be a viscosity solution of $H \leq 0$ in U. Then

$$\max_{\overline{U}}(u - \varphi) = \max_{\partial U}(u - \varphi). \tag{12}$$

Similarly, if

$$H(D^2\varphi, D\varphi, x) < 0 \text{ in } U \tag{13}$$

and $u \in LSC(\overline{U})$ is a viscosity solution of $H \geq 0$ in U then

$$\min_{\overline{U}}(u - \varphi) = \min_{\partial U}(u - \varphi). \tag{14}$$

Warning

The PDEs

$$H(D^2u, Du, u, x) = 0 \text{ and } -H(D^2u, Du, u, x) = 0$$

are **not** the same in the viscosity sense.

Exercise 3. Verify that $u(x) = -|x|$ is a viscosity solution of $|u'(x)| - 1 = 0$ on $\mathbb{R}$, but is *not* a viscosity solution of $-|u'(x)| + 1 = 0$ on $\mathbb{R}$. What is the viscosity solution of the second PDE?