Math 8590: Viscosity Solutions
Discontinuous Coefficients

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Discontinuous coefficients

$$(1)\qquad \begin{cases} H(Du) = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Definition 1 (Viscosity solution)**.** Let $f : U \to \mathbb{R}$. We say that $u \in \mathrm{USC}(\overline{U})$ is a *viscosity subsolution* of (1) if for every $x \in U$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x we have

$$H(D\varphi(x)) \leq f^*(x).$$

Similarly, we say that $u \in \mathrm{LSC}(\overline{U})$ is a *viscosity supersolution* of (1) if for every $x \in U$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x we have

$$H(D\varphi(x)) \geq f_*(x).$$

Discontinuous coefficients

Theorem 1. *Let* $U = B^0(0,1)$ *and set* $B^+ = U \cap \{x_n > 0\}$, $B^- = U \cap \{x_n < 0\}$, *and* $\Gamma = U \cap \{x_n = 0\}$. *Assume that* $f|_{B^+} \in C(\overline{B^+})$, $f|_{B^-} \in C(\overline{B^-})$ *and for all* $x \in \Gamma$

$$\lim_{B^- \ni y \to x} f(y) \leq \lim_{B^+ \ni y \to x} f(y). \tag{2}$$

Let $\varepsilon > 0$ *and let* $u, v \in C^{0,1}(\overline{U})$ *such that* $H(Du) \leq f$ *and* $H(Dv) \geq f + \varepsilon$ *in* U *in the viscosity sense of Definition 1. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v). \tag{3}$$

Proof uses auxiliary function

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha}{2}\left|x - y + \frac{1}{\sqrt{\alpha}}e_n\right|^2 \tag{4}$$

Discontinuous coefficients

A generalization:

(D) For all $x_0 \in U$ there exists $\varepsilon_{x_0} > 0$ and $\eta_{x_0} \in \mathbb{S}^{n-1}$ such that

$$f^*(x) - f_*(x + rd) \leq \omega(|x - x_0| + r), \tag{5}$$

for all $x \in U$, $r > 0$ and $d \in \mathbb{S}^{n-1}$ such that $|d - \eta_{x_0}| < \varepsilon_{x_0}$ and $x + rd \in U$, where ω is a modulus of continuity.

Theorem 2. *Let $U \subset \mathbb{R}^n$ be open and bounded, assume $f : U \to \mathbb{R}$ satisfies (D) and $H \in C(\mathbb{R}^n)$. Let $\varepsilon > 0$ and let $u, v \in C^{0,1}(\overline{U})$ such that $H(Du) \leq f$ and $H(Dv) \geq f + \varepsilon$ in U in the viscosity sense of Definition 1. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v). \tag{6}$$

We use auxiliary function

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}\left|x - y + \frac{1}{\sqrt{\alpha}}\eta_{x_0}\right|^2 - |x - x_0|^2. \tag{7}$$