Math 8590: Viscosity Solutions
Homogenization

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

Office Hours: TBD

http://www-users.math.umn.edu/~jwcalder/8590F18

Homogenization

Let $u_\varepsilon \in C(\overline{U})$ be a viscosity solution of

$$\begin{cases} u_\varepsilon + H\left(Du_\varepsilon, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{1}$$

We aim to understand u_ε as $\varepsilon \to 0^+$.

Our primary assumption is

$$\text{(Periodicity)} \quad y \mapsto H(p,y) \text{ is } \mathbb{Z}^n\text{-periodic for all } p \in \mathbb{R}^n. \tag{2}$$

We also assume that H satisfies all previous assumptions and is

$$\text{(Coercive)} \quad \liminf_{|p|\to\infty} H(p,y) > 0 \quad \text{uniformly in } y \in \mathbb{R}^n, \tag{3}$$

and

$$\text{(Nonnegative)} \quad -H(0,y) \geq 0 \quad \text{for all } y \in \mathbb{R}^n. \tag{4}$$

Homogenization

Lemma 1. *There exists a constant C such that for all $\varepsilon > 0$*

$$\|u_\varepsilon\|_{C^{0,1}(\overline{U})} \leq C. \tag{5}$$

So along a subsequence $u_\varepsilon \to u$ as $\varepsilon \to 0$, uniformly.
Suppose for x near x_0 we have the expansion

$$u_\varepsilon(x) = u(x) + \varepsilon v(\tfrac{x}{\varepsilon}) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+.$$

Then setting $y := \frac{x}{\varepsilon}$ and $p = Du(x_0)$ we find

$$H(p + Dv(y), y) = \lambda \quad \text{in } \mathbb{R}^n \tag{6}$$

for some $\lambda \in \mathbb{R}$. This is the **cell problem**.

Homogenization

Cell problem:

$$H(p + Dv(y), y) = \lambda \quad \text{in } \mathbb{R}^n \tag{7}$$

Lemma 2. *For each $p \in \mathbb{R}^n$, there exists a unique real number λ such that (7) has a $\mathbb{Z}^n$-periodic viscosity solution $v \in C^{0,1}(\mathbb{R}^n)$.*

In light of the lemma, we write

$$\overline{H}(p) := \lambda, \tag{8}$$

and the heuristics above suggest that u should be the viscosity solution of

$$u + \overline{H}(Du) = 0 \quad \text{in } U,$$

satisfying $u = 0$ on ∂U. The function $\overline{H}$ is called the **effective Hamiltonian**.

Homogenization

Theorem 1. *The sequence u_ε converges uniformly on $\overline{U}$ to the unique viscosity solution $u \in C^{0,1}(\overline{U})$ of*

$$\begin{cases} u + \overline{H}(Du) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{9}$$

The proof of Theorem 1 is based on the "perturbed test function" technique, which was pioneered in [1, 2].

References

[1] L. C. Evans. The perturbed test function method for viscosity solutions of nonlinear PDE. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 111(3-4):359–375, 1989.

[2] L. C. Evans. Periodic homogenisation of certain fully nonlinear partial differential equations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 120(3-4):245–265, 1992.