Math 8590: Viscosity Solutions
Finite difference schemes

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Finite difference approximations

Let's start with a warmup:

$$\begin{cases} u_t + cu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{1}$$

The solution is given by

$$u(x, t) = g(x - ct).$$

Characteristics are the lines $x = ct$ with speed $dx/dt = c$.

Question: How should we discretize (1)?

Finite difference approximations

$$
\begin{cases} u_t + cu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{2}
$$

Write $u_j^n \approx u(n\Delta t, j\Delta x)$ on a grid of resolution $(\Delta t, \Delta x)$. We use forward differences for u_t:

$$
u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}.
$$

For u_x we have (at least) three choices

$$
u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x}, u_x \approx \frac{u_j^n - u_{j-1}^n}{\Delta x}, \text{ or } u_x \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}.
$$

With any choice, the scheme is

$$
u_j^{n+1} = u_j^n - c\Delta t u_x.
$$

Maximum principle

When $c > 0$, upwind scheme uses backward differences for u_x:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}(u_j^n - u_{j-1}^n).$$

We can write scheme as

$$u_j^{n+1} = (1-s)\,u_j^n + s u_{j-1}^n,$$

where $s = c\Delta t/\Delta x$. If $1 - s \geq 0$, or $\Delta x/\Delta t \geq c$, this a convex combination of u_j^n and u_{j-1}^n, hence the scheme satsifies the **maximum principle**. This is the CFL stability condition $\Delta t \leq c^{-1}\Delta x$ or

$$\underbrace{\frac{\Delta x}{\Delta t}}_{\text{Numerical speed of propagation}} \geq \underbrace{c}_{\text{Speed of characteristics}}.$$

Numerical viscosity

When $c > 0$, upwind scheme uses backward differences for u_x:

$$u_j^{n+1} = u_j^n - c\Delta t \left(\frac{u_j^n - u_{j-1}^n}{\Delta x} \right).$$

We can also write scheme as

$$u_j^{n+1} = u_j^n - c\Delta t \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) + \frac{c}{2}\Delta t \Delta x \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right).$$

This looks like a discretization of

$$u_t = -cu_x + \frac{c}{2}\Delta x u_{xx}.$$

This is called **numerical viscosity**. Note $c > 0$ is essential.

Numerical viscosity

A simple nonlinear example:

$$
\begin{cases} u_t + |u_x| = 2 & \text{in } (0,1) \times (0,\infty) \\ u = g & \text{on } (0,1) \times \{t = 0\} \\ u(0) = u(1) = 0 \end{cases} \tag{3}
$$

We can think of the equation as

$$u_t + cu_x = 0$$

where $c = \text{sign}(u_x)$. Therefore, we should choose

- Backward differences for u_x when $u_x > 0$.
- Forward differences for u_x when $u_x < 0$.

This is called **upwinding**.

Finite difference schemes

We consider finite difference schems for solving the Hamilton-Jacobi equation

$$\begin{cases} H(Du, u, x) = 0 & \text{in } (0,1)^n \\ u = g & \text{on } \partial(0,1)^n. \end{cases} \tag{4}$$

Our goal is to design finite difference schemes for (4) that converge to the viscosity solution of (4) as the grid resolution tends to zero.

Notation

- For $h > 0$ let $\mathbb{Z}_h = \{hz \,:\, z \in \mathbb{Z}\}$ and $\mathbb{Z}_h^n = (\mathbb{Z}_h)^n$.
- For a set $\mathcal{O} \subset \mathbb{R}^n$ we define $\mathcal{O}_h := \mathcal{O} \cap \mathbb{Z}_h^n$, and $\partial O_h := (\partial O) \cap \mathbb{Z}_h^n$.
- We will always assume that $1/h$ is an integer.
- Given a function $u : [0,1]_n^h \to \mathbb{R}$, we define the forward and backward difference quotients by

$$\nabla_i^{\pm} u(x) := \pm \frac{u(x \pm he_i) - u(x)}{h}, \tag{5}$$

and we set

$$\nabla^{\pm} u(x) = (\nabla_1^{\pm} u(x), \ldots, \nabla_n^{\pm} u(x)).$$

Basic example

Exercise 1. Consider the following finite difference scheme for the one dimensional eikonal equation

$$|\nabla_1^+ u_h(x)| = 1 \quad \text{for } x \in [0,1)_h, \quad \text{and} \quad u_h(0) = u_h(1) = 0. \tag{6}$$

Show that the scheme is not well-posed, that is, depending on whether $1/h$ is even or odd, there is either no solution, or there is more than one solution.

Hamilton-Jacobi-Bellman Equation

Recall the Hamilton-Jacobi-Bellman equation

$$H(Du, x) = 0$$

where

$$H(p, x) = \sup_{|a|=1} \left\{-p \cdot a - L(a, x)\right\}.$$

In this case, the solution u satisfies the dynamic programming principle

$$u(x) = \inf_{y \in \partial B(x,r)} \{u(y) + T(x, y)\}.$$

The infimum on the right is attained at some $y \in \partial B(x, r)$ so we have

$$u(x) = u(y) + T(x, y).$$

Key observation: $u(x)$ depends only on $u(y)$ with $u(y) \leq u(x)$.

Basic monotone scheme

We define the *monotone* finite differences

$$\nabla_i^{\mathbf{m}} u = \mathbf{m}(\nabla_i^+ u, \nabla_i^- u), \tag{7}$$

where

$$\mathbf{m}(a,b) = \begin{cases} a, & \text{if } a+b<0 \text{ and } a \le 0 \\ b, & \text{if } a+b \ge 0 \text{ and } b \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

We also define the *monotone gradient* by

$$\nabla^{\mathbf{m}} u = (\nabla_1^{\mathbf{m}} u, \ldots, \nabla_n^{\mathbf{m}} u).$$

Key property:

$$|\nabla_i^{\mathbf{m}} u(x)| = \frac{1}{h} \max \left\{ u(x) - u(x+he_i), u(x) - u(x-he_i), 0 \right\}.$$

Basic monotone scheme

Exercise 2. Consider the following monotone finite difference scheme for the one dimensional eikonal equation:

$$|\nabla_1^{\mathbf{m}} u_h(x)| = 1 \quad \text{for } x \in (0,1)_h, \quad \text{and } u_h(0) = u_h(1) = 0.$$

Find the solution u_h explicitly, and show that $u_h \to \frac{1}{2} - |x|$ as $h \to 0^+$.

Back to maximum principle

Proposition 1. *If* $u(x) = v(x)$ *and* $u \leq v$ *then*

$$|\nabla_i^{\mathbf{m}} u(x)| \geq |\nabla_i^{\mathbf{m}} v(x)| \quad \textit{for all } i.$$

Lemma 1. *Suppose* H *is given by*

$$H(p, x) = \sup_{|a|=1} \{-p \cdot a - L(a, x)\}$$

and L *satisfies*

$$L(a_1, \ldots, a_n, x) = L(|a_1|, \ldots, |a_n|, x) \;\; \textit{for all } x. \tag{8}$$

If $u(x) = v(x)$ *and* $u \leq v$ *then*

$$H(\nabla^{\mathbf{m}} u(x), x) \geq H(\nabla^{\mathbf{m}} v(x), x). \tag{9}$$

Back to maximum principle

A general finite difference scheme has the form

$$\begin{cases} S_h(u_h, u_h(x), x) = 0 & \text{in } (0,1)^n_h \\ u_h = g & \text{on } \partial(0,1)^n_h, \end{cases} \tag{10}$$

where

$$S_h : X_h \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

and X_h denotes the collection of real-valued functions on $[0,1]^n_h$.

Definition 1. We say the scheme S_h is *monotone* if

$$u \leq v \implies S_h(u,t,x) \geq S_h(v,t,x) \tag{11}$$

for all $u, v \in X_h$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Note: Equation (10) can be **any** approximation scheme satisfying (11).

Barles-Souganidis framework [1]

Every **monotone**, **consitent**, and **stable** scheme converges to the viscosity solution, provided the PDE is well-posed.

- **Well-posed** here means the PDE satisfies a comparison principle with boundary conditions in the **viscosity sense** (called strong uniqueness in [1]).

Boundary conditions in the viscosity sense

$$
\begin{cases} H(Du, u, x) = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{12}
$$

Definition 2. We say $u \in \mathrm{USC}(\overline{U})$ is a *viscosity subsolution* of (12) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x with respect to $\overline{U}$

$$
\begin{cases} H(D\varphi(x), u(x), x) \leq 0, & \text{if } x \in U \\ \min\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \leq 0 & \text{if } x \in \partial U. \end{cases}
$$

Likewise, we say that $u \in \mathrm{LSC}(\overline{U})$ is a *viscosity supersolution* of (12) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x with respect to $\overline{U}$

$$\begin{cases} H(D\varphi(x), u(x), x) \geq 0, & \text{if } x \in U \\ \max\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \geq 0 & \text{if } x \in \partial U. \end{cases}$$

Finally, we say that u is a *viscosity solution* of (12) if u is both a viscosity sub- and supersolution. In this case, we say that the boundary conditions in (12) hold in the *viscosity sense*

Strong uniqueness means if $u \in \mathrm{USC}(\overline{U})$ is a subsolution (as above), and $v \in \mathrm{LSC}(\overline{U})$ is a supersolution, then $u \leq v$ on $\overline{U}$.

Barles-Souganidis framework [1]

Definition 3. We say the scheme S_h is *monotone* if

$$u \leq v \implies S_h(u,t,x) \geq S_h(v,t,x) \tag{13}$$

for all $u, v \in X_h$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Definition 4. We say the scheme S_h is *consistent* if

$$\lim_{\substack{y \to x \\ h \to 0^+ \\ \gamma \to 0}} S_h(\varphi + \gamma, \varphi(y) + \gamma, y) = H(D\varphi(x), \varphi(x), x) \tag{14}$$

for all $\varphi \in C^\infty(\mathbb{R}^n)$.

Definition 5. We say the scheme S_h is *stable* if the solutions u_h are uniformly bounded as $h \to 0^+$, that is, there exists $C > 0$ such that

$$\sup_{h>0} \sup_{x \in [0,1]^n_h} |u_h(x)| \leq C.$$

Barles-Souganidis framework [1]

$$\begin{cases} H(Du, u, x) = 0 & \text{in } (0,1)^n \\ u = g & \text{on } \partial(0,1)^n. \end{cases} \tag{15}$$

$$\begin{cases} S_h(u_h, u_h(x), x) = 0 & \text{in } (0,1)_h^n \\ u_h = g & \text{on } \partial(0,1)_h^n, \end{cases} \tag{16}$$

Theorem 1. *Suppose* (15) *enjoys strong uniqueness, and* S_h *is monotone, consistent, and stable. Then* $u_h \to u$ *uniformly on* $[0,1]^n$ *as* $h \to 0^+$, *where* u *is the unique viscosity solution of* (4).

Monotone schemes are first order (at best)

Write our monotone scheme as

$$F[u](x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \ldots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x),$$

where $F = F(a_1, \ldots, a_{2n}, z, x)$. Recall from HW that F is monotone if and only if F is nondecreasing in each a_i, i.e., $F_{a_i} \geq 0$ for all i. Let $M > 0$ and define

$$\mathcal{S}_M := \left\{ \varphi \in C^\infty(\mathbb{R}^n) \,:\, \|\varphi\|_{C^3(\mathbb{R}^n)} \leq M \right\}.$$

We define the local truncation error by

$$\operatorname{err}(M, h) := \sup_{\substack{\varphi \in \mathcal{S}_M \\ x \in [0,1]^n}} |F[\varphi](x) - H(D\varphi(x), \varphi(x), x)|.$$

Monotone schemes are first order (at best)

Theorem 2. *Let F be monotone and smooth, and assume H is smooth. Suppose that for some $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in [0,1]^n$, and $i \in \{1, \ldots, n\}$*

$$H_{p_i}(p, z, x) \neq 0. \tag{17}$$

Then there exists $M > 0$, $C > 0$, $c > 0$ and $\overline{h} > 0$ such that for all $0 < h < \overline{h}$

$$ch \leq err(M, h) \leq Ch. \tag{18}$$

Note: In this case, consistency of the scheme states that

$$F(p_1, -p_1, \ldots, p_n, -p_n, z, x) = H(p, z, x). \tag{19}$$

Convergence rates

Let $u \in C^{0,1}([0,1]^n)$ be the unique viscosity solution of

(20) $$\begin{cases} H(Du, x) = 0 & \text{in } (0,1)^n \\ u = 0 & \text{on } \partial(0,1)^n, \end{cases}$$

and consider the monotone finite difference scheme

(21) $$\begin{cases} H(\nabla^{\mathbf{m}} u_h(x), x) = 0 & \text{in } (0,1)^n_h \\ u_h = 0 & \text{on } \partial(0,1)^n_h. \end{cases}$$

We consider the Hamilton-Jacobi-Bellman equation where

(22) $$H(p,x) = \sup_{|a|=1} \left\{ -p \cdot a - L(a,x) \right\}.$$

We assume L is Lipschitz and satisfies all prior assumptions.

Convergence rates

We first consider existence/uniqueness of solutions to our scheme.

$$\begin{cases} H(\nabla^{\mathbf{m}} u_h(x), x) = 0 & \text{in } (0,1)^n_h \\ u_h = 0 & \text{on } \partial(0,1)^n_h. \end{cases} \tag{23}$$

We use the Perron method.

Definition 6. We say that $u_h : [0,1]^n_h \to \mathbb{R}$ is a *subsolution* of (23) if $H(\nabla^{\mathbf{m}} u_h, x) \leq 0$ in $(0,1)^n_h$ and $u_h \leq 0$ on $\partial(0,1)^n_h$. We define supersolutions analogously.

Lemma 2. *If u and v are sub- and supersolutions of* (21)*, respectively, then $u \leq v$ on $[0,1]^n_h$.*

Lemma 3. *There exists a unique grid function $u_h : [0,1]^n_h \to \mathbb{R}$ satisfying the monotone scheme* (21)*. Furthermore, the sequence u_h is nonnegative and uniformly bounded.*

Convergence rates

Proposition 2. *The Hamiltonian H is Lipschitz continuous.*

Theorem 3. *There exists a constant $C > 0$ such that*

$$|u - u_h| \leq C\sqrt{h}.$$

References

[1] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic analysis*, 4(3):271–283, 1991.